

XI

THE LORENTZ TRANSFORMATION

THE results of the last three sections show that the apparent incompatibility of the law of propagation of light with the principle of relativity (Section VII) has been derived by means of a consideration which borrowed two unjustifiable hypotheses from classical mechanics; these are as follows:

- (1) The time-interval (time) between two events is independent of the condition of motion of the body of reference.
- (2) The space-interval (distance) between two points of a rigid body is independent of the condition of motion of the body of reference.

If we drop these hypotheses, then the dilemma of Section VII disappears, because the theorem of the addition of velocities derived in Section VI becomes invalid. The possibility presents itself that the law of the propagation of light *in vacuo* may be compatible with the principle of relativity, and the question arises: How have we to modify the considerations of Section VI in order to remove

the apparent disagreement between these two fundamental results of experience? This question leads to a general one. In the discussion of Section VI we have to do with places and times relative both to the train and to the embankment. How are we to find the place and time of an event in relation to the train, when we know the place and time of the event with respect to the railway embankment? Is there a thinkable answer to this question of such a nature that the law of transmission of light *in vacuo* does not contradict the principle of relativity? In other words: Can we conceive of a relation between place and time of the individual events relative to both reference-bodies, such that every ray of light possesses the velocity of transmission c relative to the embankment and relative to the train? This question leads to a quite definite positive answer, and to a perfectly definite transformation law for the space-time magnitudes of an event when changing over from one body of reference to another.

Before we deal with this, we shall introduce the following incidental consideration. Up to the present we have only considered events taking place along the embankment, which had mathematically to assume the function of a straight line. In the manner indicated in Section II we can imagine this reference-body supplemented laterally and in a vertical direction by means of a

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framework of rods, so that an event which takes place anywhere can be localised with reference to this framework. Similarly, we can imagine the train travelling with the velocity v to be continued across the whole of space, so that every event, no matter how far off it may be, could also be localised with respect to the second framework. Without committing any fundamental error, we can disregard the fact that in reality these frameworks would continually interfere with each other, owing to the impenetrability of solid bodies. In every such framework we imagine three surfaces perpendicular to each other marked out, and designated as "co-ordinate planes" ("co-ordinate system"). A co-ordinate system K then corresponds to the embankment, and a co-ordinate system K' to the train. An event, wherever it may have taken place, would be fixed in space with respect to K by the three perpendiculars x, y, z on the co-ordinate planes, and with regard to time by a time-value t . Relative to K' , *the same event* would be fixed in respect of space and time by corresponding values x', y', z', t' , which of course are not identical with x, y, z, t . It has already been set forth in detail how these magnitudes are to be regarded as results of physical measurements.

Obviously our problem can be exactly formulated in the following manner. What are the

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values x', y', z', t' of an event with respect to K' , when the magnitudes x, y, z, t , of the same event with respect to K are given? The relations must be so chosen that the law of the transmission of light *in vacuo* is satisfied for one and the same ray of light (and of course for every ray) with respect to K and K' . For the relative orientation in space of the co-ordinate systems indicated in the diagram (Fig. 2), this problem is solved by means of the equations:

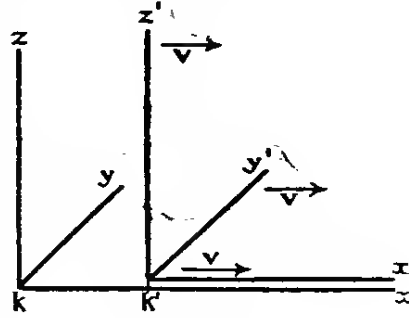


FIG. 2.

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v}{c^2} \cdot x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This system of equations is known as the “Lorentz transformation.”¹

If in place of the law of transmission of light we had taken as our basis the tacit assumptions of the older mechanics as to the absolute character

¹ A simple derivation of the Lorentz transformation is given in Appendix I.

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of times and lengths, then instead of the above we should have obtained the following equations:

$$\begin{aligned}x' &= x - vt \\y' &= y \\z' &= z \\t' &= t.\end{aligned}$$

This system of equations is often termed the “Galilei transformation.” The Galilei transformation can be obtained from the Lorentz transformation by substituting an infinitely large value for the velocity of light c in the latter transformation.

Aided by the following illustration, we can readily see that, in accordance with the Lorentz transformation, the law of the transmission of light *in vacuo* is satisfied both for the reference-body K and for the reference-body K' . A light-signal is sent along the positive x -axis, and this light-stimulus advances in accordance with the equation

$$x = ct,$$

i.e. with the velocity c . According to the equations of the Lorentz transformation, this simple relation between x and t involves a relation between x' and t' . In point of fact, if we substitute for x the value ct in the first and fourth equations of the Lorentz transformation, we obtain:

$$x' = \frac{(c - v)t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t' = \frac{\left(1 - \frac{v}{c}\right)t}{\sqrt{1 - \frac{v^2}{c^2}}},$$

from which, by division, the expression

$$x' = ct'$$

immediately follows. If referred to the system K' , the propagation of light takes place according to this equation. We thus see that the velocity of transmission relative to the reference-body K' is also equal to c . The same result is obtained for rays of light advancing in any other direction whatsoever. Of course this is not surprising, since the equations of the Lorentz transformation were derived conformably to this point of view.