Chapter 12

Electromagnetic Waves

12.1 INTRODUCTION

In the previous chapters, we learned about the production of electric and magnetic fields in space, due to charges and due to currents, respectively. We then learned that an electric field would also be produced by a changing magnetic field. In this chapter, we will show that these laws are not yet complete, and require the addition of one further new concept, the concept of displacement current. With the addition of this concept, we will then summarize the laws for the production of electric and magnetic fields in the form of the four Maxwell equations. In turn this will lead to the use of these equations to predict the existence and the properties of electromagnetic waves, which we will then discuss.

12.2 DISPLACEMENT CURRENT

In Chap. 7, we learned about the production of a magnetic field in space, due to currents in a wire. The law that we developed that must be followed is **Ampere's law**, which states that when we go around a closed loop (such as a circle) and add together the component of the magnetic field along the loop times Δl (the infinitesimal length along the loop), i.e. $\sum B \cos \theta \Delta l$, this sum equals μ_0 times the current through the closed loop. Let us analyze this current in more detail. Consider the circular loop a in Fig. 12-1 and a wire, w, with current I perpendicular to the loop. What do we mean by the current flowing *through* the loop a? A possible answer is that the current is that which flows through the area A_1 (shown in the figure) which is bounded by the loop. Since in our case the only current is in the wire, which passes through A_1, the current through the loop is I. A_1, however, is not the only area bounded by the loop a. For instance, consider the cylindrical shaped surface formed by disk surface A_2 and cylinder surface A_3. The combination of these two surfaces is bounded by loop a, just as area A_1 is. Indeed it is easy to see that there are an infinite number of surfaces bound by the loop a. This is generally true for any closed loop whether circular or not, whether in a plane or not. For the case of our loop a and our $A_2 + A_3$ combined surface bound by a we can see that there is no current flowing through area A_3, and the current flowing through A_2 is just the current in the wire, I. We therefore get the same current whether we use A_1 for the area, or the $(A_2 + A_3)$ surface for the area. It would therefore seem that there is no ambiguity about which surface bounded by the loop one should use. This is generally true as long as the wires don't end, and, if they do end, then we know that for DC currents, there can be no current in such a wire and therefore no problem either. However, we do run into a problem if the wires terminate, and we have AC currents. Consider, for instance, a circuit with a

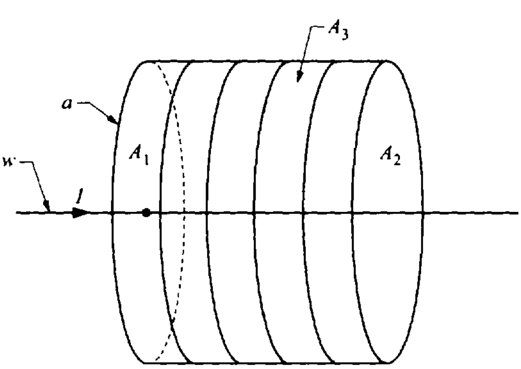

Fig. 12-1

wire entering one plate of a capacitor. During the time that the capacitor is being charged, a current flows in the wire but no current flows from one capacitor plate to the other. This creates a situation in which a closed loop can bound one surface with current flowing through it and bound another surface with no current through it. This can be seen for the case depicted in Fig. 12-2. Here a wire w is carrying a current I to the capacitor plate during the time that the capacitor is being charged. We use the circle a as the closed loop and consider $\sum (B \cos \theta \, \Delta l)$ around the loop. By Ampere's law, this should equal the current through the area bounded by the curve. But the answer we get is quite different if we use the area A_1, or the area $A_2 + A_3$. For area A_1, the current is I, the current in the wire, which flows through A_1. For the area $A_2 + A_3$, we would get zero as the answer, since no current is flowing between the capacitor plates. We are thus faced with a dilemma, since we get contradictory answers for different surfaces bounded by the same loop. Which answer should we use in Ampere's law? It would be nice if the current through any surface bounded by the loop were the same. The capacitor plate situation affords the opportunity to accomplish this by broadening the definition of current for Ampere's law. We use the term conduction current, which is current conducted in a wire or some conducting medium, to distinguish it from the displacement current which we shall shortly define. Even though there is no conduction current between the capacitor plates, there is something there that is not present at area A_1. That something is a changing electric field. Perhaps this changing electric field can be associated with a new "displacement" current that contributes to Ampere's law in just such a way that the "current" through A_2 equals the current through A_1 in Fig. 12-2. Let us calculate how this could happen.

We know that the field within a parallel plate capacitor is uniform and is equal to $E = q/\varepsilon_0 A$, where q is the charge on the capacitor and A is the area of the capacitor. Strictly speaking, this is only true for the field within large plates, and the field varies as one approaches the edge of the plates. However, a more exact calculation shows that this won't change our conclusions. If the capacitor is being charged, then both q and E are changing, and we can write that $\Delta E/\Delta t = (\Delta q/\Delta t)/\varepsilon_0 A = I/\varepsilon_0 A$. Thus,

$$I = \varepsilon_0 A(\Delta E/\Delta t) \tag{12.1a}$$

If we define a "**displacement current density**" as

$$J_D = \varepsilon_0(\Delta E/\Delta t) \tag{12.1b}$$

then we find that the current through surface A_2 is $I_D = J_D A = I$, the same result as for area A_1. The **displacement current** I_D can also be written as $I_D = \varepsilon_0(\Delta EA/\Delta t) = \varepsilon_0 \Delta\Psi/\Delta t$, where Ψ is the electric flux through the area. By modifying Ampere's law to include displacement current, we eliminate the contradiction that we had previously discussed. Ampere's law would then state that

$$\sum B \cos \theta \, \Delta l = \mu_0(I + I_D) \tag{12.2}$$

In the case of our wire and capacitor, $I_D = 0$ through A_1, while $I = 0$ through A_2. While Eq. (12.2) eliminates our ambiguity, we must still demonstrate that it is true. We must find if this concept of displacement current predicts something new, and then test this prediction experimentally.

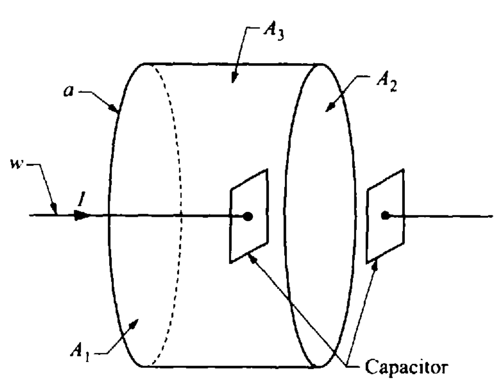

Fig. 12-2

Problem 12.1. Using Ampere's law, calculate the magnetic field between circular parallel plate capacitors, at a distance r from the center, when the capacitor is being charged at the rate of I coulombs/second.

Solution

In Fig. 12-3, we draw the circular capacitor plates which are being charged. At a distance, r, from the center, we draw a circular path that we will use for Ampere's law. The electric field points to the right if the left plate is positively charged, and that charge is being increased at the rate of $I = \Delta q/\Delta t$ C/s. The electric field is uniform, and equal to $q/\varepsilon_0 A$, and increasing at the rate of $\Delta E/\Delta t = (\Delta q/\Delta t)/\varepsilon_0 A = I/\varepsilon_0 A$, where A is the area of the plates. The displacement current density is equal to $\varepsilon_0 \Delta E/\Delta t$, and is uniform within the region. The magnetic field is the same at every point on the circular path, by symmetry. We go around the path in the direction shown, using the right-hand rule with our thumb in the direction of **E**, so our fingers curl about the path in this direction. Adding the magnetic field along the curve we get $\sum B \cos\theta\, \Delta l = B(2\pi r)$. We must now calculate the total current going through this loop. There is no conduction current, so the only contribution comes from the displacement current. The total displacement current through this curve is $J_D A_r = (\varepsilon_0 \Delta E/\Delta t)(\pi r^2) = \varepsilon_0(I/\varepsilon_0 A)\pi r^2 = I(\pi r^2/A)$. By Ampere's law, we therefore have that $B(2\pi r) = \mu_0 I(\pi r^2/A)$, or $B = \mu_0 \pi I r/A$. This shows that the field increases linearly with r as one moves from the center to the edge of the plate. This can, of course, be tested experimentally and the result agrees with prediction.

Outside the capacitor plates, we can also calculate the field using Ampere's law. In that case, the displacement current density would be $(\varepsilon_0 \Delta E/\Delta t)$ within the plates, and zero outside the plates. The total current through the area would then be $J_D A = (\varepsilon_0 \Delta E/\Delta t)A = I$, giving $B(2\pi r) = \mu_0 I$, or $B = \mu_0 I/2\pi r$, as for the field of a long straight wire.

Problem 12.2. A rectangle *abcd*, with sides of 60 cm × 80 cm, is in an electric field of 10^3 V/m directed into the paper, as in Fig. 12-4. The field is increasing at the rate of 300 V/m · s.

(*a*) What is the displacement current through this area?

(*b*) What is the direction of the component of the magnetic field along *ab*?

Solution

(*a*) The displacement current density is $J_D = \varepsilon_0 \Delta E/\Delta t$, and is constant within the area. Since the field is increasing, $\Delta E/\Delta t$ is also into the paper, as is the direction of J_D. The displacement current is therefore $J_D A = \varepsilon_0 A\, \Delta E/\Delta t = 8.85 \times 10^{-12}(0.48)(3 \times 10^2) = 1.27 \times 10^{-9}$ A, into the paper.

(*b*) Since I_D is into the paper, the positive direction for going around *abcd* is clockwise ($a \to b \to c \to d$). Ampere's law states that $\sum B \cos\theta\, \Delta l = +\mu_0(I + I_D) = \mu_0 I_D$, in this case. Therefore, the components of B are in the positive direction of circling around the rectangle, and along *ab* the field is directed from *a* to *b*.

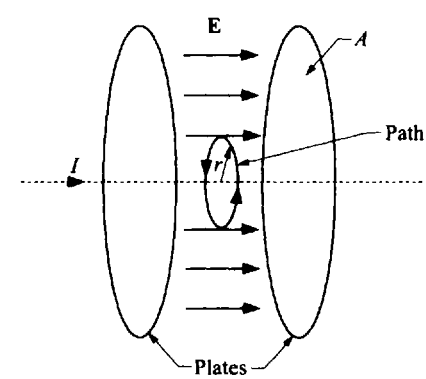

Fig. 12-3

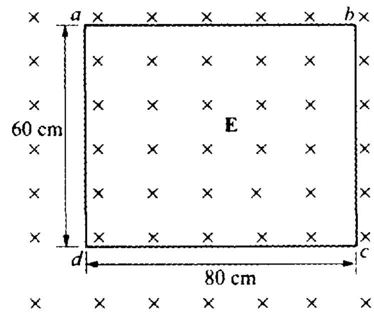

Fig. 12-4

Problem 12.3. For the same case as in Problem 12.2, what would the answers be if the rectangle were located in a region of dielectric constant 3.5?

Solution

Using the same reasoning that we used in defining the displacement current originally, but for a capacitor filled with a dielectric material of dielectric constant κ, we would define the displacement current in this case as $J_D = \varepsilon \Delta E/\Delta t$, where $\varepsilon = \kappa\varepsilon_0$. Thus we need only replace ε_0 by ε to get the answer for our problem. The current is therefore $1.27 \times 10^{-9}(3.5) = 4.45 \times 10^{-9}$ A, and the direction is the same as for Problem 12.2.

In order to determine whether the concept of displacement current is generally valid, we seek some prediction which is different with or without this concept. This is what Maxwell did after summarizing the laws of electricity and magnetism in the famous **Maxwell equations**.

12.3 MAXWELL'S EQUATIONS

We have already stated that, after including the displacement current, it is possible to summarize the laws governing the creation of electrical and magnetic fields in four fundamental equations.

The first equation is **Gauss' law**, which states that electric fields can be established by free charges. This law is written in terms of the electric flux that passes through a closed surface, and depends on the understanding that all electric field lines start at positive charges and end on negative charges (lines can also go to infinity, such as those of an isolated point charge, where they are presumed to land on opposite charges at that distance). By convention the number of electric field lines per unit area, the electric flux density, at a given point is chosen equal to the magnitude of the electric field at that point. As discussed in Chap. 3 it then equals the electric field at every other point as well. Gauss' law then relates the total charge within a closed surface to the net number of electric field lines that pass through the surface. Gauss' law can be written as

$$\sum \Psi_E = \sum Q/\varepsilon_0 \tag{12.3}$$

where Ψ_E = flux through an infinitesimal surface area $A = E(\cos\theta)A$ with θ the angle below **E** and the outward normal to the surface element A, and the sum goes over all elements A making up the closed surface. The sum over charges includes all charges within the closed surface.

The second (Maxwell) equation is based on how this same concept applies to magnetic fields. We have learned that there are no magnetic monopoles that act as sources for a magnetic field. Therefore magnetic fields do not have poles where they begin or end. All magnetic field lines must therefore close on themselves. This means that any magnetic field line that passes through a closed surface must necessarily pass through the surface again in the opposite direction, in order to close on itself. This means that the net total magnetic flux which passes through a surface is zero. This is written as

$$\sum \Phi_B = 0 \tag{12.4}$$

where Φ_B = (magnetic flux through an infinitesimal surface area A) = $B(\cos\theta)A$ with θ the angle between **B** and the outward normal to the surface element A, and the sum goes over all elements A making up the closed surface.

The third equation states that an electric field can also be produced by a changing magnetic flux. This is **Faraday's law**, which can be written as

$$\sum E(\cos\theta)\Delta l = -\Delta\Phi_B/\Delta t \tag{12.5}$$

where Δl is an infinitesimal length along a closed curve, θ is the angle between **E** and the tangent to the curve at Δl and the sum is taken over all the elements Δl of the closed curve, and the flux Φ_B is the total magnetic flux through the area bounded by the curve.

The fourth, and last equation, states that magnetic fields are created by currents, either conduction current or displacement current. This can be written in the form of **Ampere's law**, including displacement current, as

$$\sum B(\cos\theta)\Delta l = \mu_0(I + I_D) = \mu_0(I + \varepsilon_0\,\Delta\Psi_E/\Delta t) \tag{12.6}$$

where Δl is again an infinitesimal length along a closed curve, θ is the angle between **B** and the tangent to the curve at Δl, and the sum is taken over all elements of the closed curve. The current I is the total current passing through a surface bounded by the curve, and Ψ_E is the total electric flux through the same surface.

These four equations are relationships between the electric and magnetic fields and their sources, charges and currents. The electric and magnetic fluxes are determined directly from the electric and magnetic fields and are not separate variables. Thus these equations tell us how to calculate the electric and magnetic fields that are produced by charges, both at rest and moving. The particular form that we have used for these equations is not the most useful for actual calculations, but is the easiest to understand conceptually. For purposes of calculations, these equations are expressed more formally in the language of the integral and differential calculus, which can then be solved for specific cases.

We have written these equations for the case of free space, and not for the situation in which there is dielectric or magnetic material present. For the case of materials, one must modify these equations using the concepts of **electric displacement** (D) and **magnetic intensity** (H). We will not write down these modified equations, but will point out where changes occur in the solutions that we will discuss.

These four equations constitute Maxwell's equations, which are the fundamental laws governing the existence of electric and magnetic fields, which are jointly called electromagnetic fields. We see clearly from these equations that electric and magnetic fields are not really independent quantities, but are rather quantities that are bound together by these relationships. Changes in one produce or modify the other. These equations are remarkable in that, unlike Newton's laws, they do not require fundamental modification as a result of the theory of relativity. In fact, the solutions to these equations gave rise to questions that required the theory of relativity for their resolution. Furthermore, the quantum theory also accepts these equations as the fundamental ones describing electromagnetic phenomena, requiring only a proper interpretation in light of the fundamentally new concepts of quantum mechanics.

In order to give a complete description, in theory, of electromagnetic phenomena, we must add the laws that tell us what effect these fields have on objects. This is given by the statement that electric fields exert forces on any electrical charges, while magnetic fields exert forces on moving charges. The magnitudes and directions of these forces were discussed previously in Chaps. 3 and 6.

12.4 ELECTROMAGNETIC WAVES

Maxwell was able to show that there were solutions to these equations that corresponded to waves propagating in free space, i.e. in regions where there are no charges or currents. These waves, which he called **electromagnetic waves** (EM) had special properties, which could be derived from these equations. In all the waves previously discussed the wave was a consequence of the vibration of molecules of a medium about their equilibrium positions—their displacement—and the propagation of this disturbance with a velocity characteristic of the medium. In the case of electromagnetic waves the time varying quantity is not the displacement but rather the electric and magnetic fields at a point in space. Indeed, for electromagnetic waves one does not even need a medium—they can travel through empty space. Like ordinary waves, however, they do have a characteristic velocity which depends on the material the wave travels through and has an especially significant value in empty space. Maxwell was able to show that these waves were transverse, and that their speed, in free space was equal to $1/\sqrt{\varepsilon_0 \mu_0} = c$.

For a wave traveling in the x direction, this means that the electric and magnetic fields associated with this wave are in the y–z plane. Indeed, one can show that these fields are also perpendicular to each other, and that their magnitudes are given by:

$$E = cB \tag{12.7}$$

It is important to note that these results would not be true if one left out the term for displacement current. That EM waves exist is the strongest evidence that the displacement current should be included in Ampere's law. Furthermore, Maxwell's prediction that these electromagnetic waves travel with speed $1/\sqrt{\mu_0 \varepsilon_0}$, which numerically equals 3×10^8 m/s, matches the measured value for the **speed of light**. This quickly led to the realization that light consists of electromagnetic waves in a certain frequency range to which the eye is sensitive and can "see". Approximately 22 years after Maxwell predicted the existence of these electromagnetic waves, and delineated their properties, Henry produced and detected these waves in the radio range of frequencies.

If the medium in which this wave propagates is not free space, but rather a material with a **dielectric constant** κ and **magnetic permeability** κ_M, then the velocity of the waves will become

$$v = 1/\sqrt{\kappa\kappa_\mu \mu_0 \varepsilon_0} = 1/\sqrt{\mu\varepsilon}. \tag{12.8}$$

Problem 12.4. For an electromagnetic wave, traveling in a medium with dielectric constant 3.5, and magnetic permeability 1.2, what is the speed of this wave?

Solution

Using Eq. (*12.8*), and knowing that $1/\sqrt{\mu_0 \varepsilon_0} = 3 \times 10^8$ m/s, we get $v = 3 \times 10^8/\sqrt{(3.5)(1.2)} = 1.46 \times 10^8$ m/s.

Problem 12.5. For an electromagnetic wave, traveling in the x direction in free space, the electric field has a magnitude of 1.5 V/m, and is in the y direction. What is the magnitude and direction of the magnetic field?

Solution

Using Eq. (*12.7*), the magnitude of B is $B = E/c = (1.5\ \text{V/m})/(3 \times 10^8\ \text{m/s}) = 0.5 \times 10^{-8}$ T. The direction of the magnetic field is in the $+z$ direction, perpendicular to both **B** and the direction of propagation.

Problem 12.6. For an electromagnetic wave, traveling in the $-x$ direction in free space, the electric field has a magnitude of 1.5 V/m, and is in the $+y$ direction. What is the magnitude and direction of the magnetic field?

Solution

Using Eq. (*12.7*), the magnitude of B is $B = E/c = (1.5\ \text{V/m})/(3 \times 10^8\ \text{m/s}) = 0.5 \times 10^{-8}$ T. To get the direction we note that it can be shown that, in general, the three perpendicular directions, **E**, **B** and **c**, are related like **v**, **B** and **F** in the magnetic force on a charge. This means that if your fingers are in the direction that rotates **E** into **B**, the thumb will point in the direction of the velocity, **c**. Therefore, the direction of **B** in our problem is in $-z$.

Problem 12.7. For an electromagnetic wave, traveling in the $+x$ direction in free space, the electric field has a magnitude of 1.5 V/m, and is in the $+z$ direction. What is the magnitude and direction of the magnetic field?

Solution

Using Eq. (*12.7*), the magnitude of B is $B = E/c = 1.5/3 \times 10^8 = 0.5 \times 10^{-8}$ T. We know that the magnetic field must be perpendicular to both **E** and the velocity, **c**, and therefore is in the $\pm y$ direction. We showed in Problem 12.6, that the three perpendicular directions, **E**, **B** and **c**, are related so that if your fingers are in the direction that rotates **E** into **B**, the thumb will point in the direction of the velocity, **c**. Applying this to our case, we see that the magnetic field is in the $-y$ direction.

As noted, it can be demonstrated that light is one form of electromagnetic wave. The fact that a light wave travels with the speed predicted for an electromagnetic wave is one of the reasons that it was quickly suspected that this was true; given Maxwell's theoretical result. How can one measure the speed of light? One method is illustrated in the next problem.

Problem 12.8. Light passes through an opening in a rim of a notched wheel, as in Fig. 12-5. Light travels to a mirror at a distance of 5×10^4 m and is reflected back to the wheel. There are 50 notches in the wheel. At what angular speed must the wheel turn so that the reflected light passes through the adjacent opening?

Solution

The light that passes through one notch travels to the mirror and then back to the wheel in a time equal to $2L/c$, where L is the distance to the mirror. During this same time, the wheel has to turn just far enough that the next notch is now in the position of the first notch. Since there are 50 notches on the wheel, the wheel has to rotate through 1/50 of a full rotation, or through an angle of $2\pi/50$. The time that this takes is $(2\pi/50)/\omega$. Therefore $(2\pi/50)/\omega = 2L/c$, or $\omega = (2\pi/50)c/2L = 2\pi(3 \times 10^8)/(50)(2)(5 \times 10^4) = 377$ rad/s.

Just as in the case of sound waves, it is useful to consider electromagnetic waves that are sinusoidal. This means that if we take a picture of the wave at any time, the disturbance will vary sinusoidally in space along the direction of propagation. Furthermore, at any position is space, the disturbance will vary sinusoidally in time. As in the case of sound, one can have different shaped electromagnetic waves, such as spherical waves emitting from a local region, but far away they appear nearly planar over a region small compared to the distance from the disturbance. In such a region the light wave has the same value of electric and magnetic field at all points in the plane perpendicular to the direction of propagation, and varying in lock step. It is important to keep in mind that the disturbance associated with an electromagnetic wave is the electric and magnetic field along the wave. In Fig. 12-6, we show a

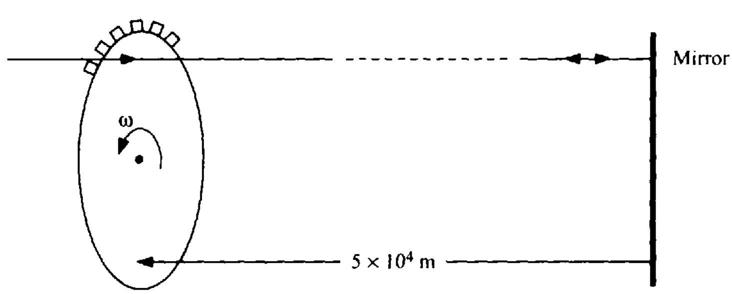

Fig. 12-5

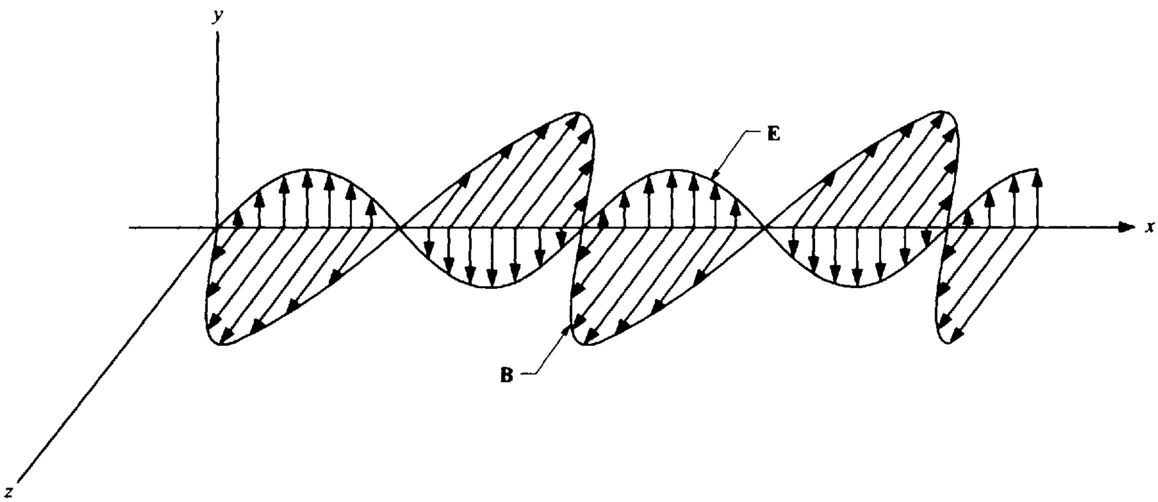

Fig. 12-6

plane wave as it varies in space at a particular instant of time. The wave is traveling in the positive x direction, and both the electric and magnetic fields are transverse to this direction. We draw the electric field in the y direction, and then the magnetic field must be in the z direction as we showed in the previous problems. This wave is said to be linearly polarized in the y direction, which is the direction of the electric field. The electric and magnetic fields have magnitudes that are related by $E = cB$ at every point, and the wave travels with a velocity c. The distance between successive crests or between successive troughs is the wavelength, λ, of the wave. The time that it takes for the wave to travel a distance of one wavelength is the period, T, of the wave, which is also the time for the wave to go from crest to crest at any given point in space. As with all waves, this leads to the relationship that $c = \lambda/T = \lambda f$, where f is the frequency.

Electromagnetic waves exist with wavelengths ranging from very small to very large (and corresponding frequencies from very large to very small). The various possible wavelength (and frequency) ranges constitute the electromagnetic spectrum. For small frequencies the wave is usually denoted by its frequency, and for short wavelength it is denoted by its wavelength. In Table 12.1, we list the frequency range for some common types of electromagnetic waves with small frequencies, and others with small wavelengths. In the next problem, we will complete the table.

Problem 12.9. For the electromagnetic waves in the table, calculate the missing wavelengths and frequencies to complete the table.

Table 12.1 Frequency Range of Common Types of Electromagnetic Waves

Type of wave	Frequency	Wavelength (m)
Power line	60	
AM radio	$(0.5\text{–}1.5) \times 10^6$	
FM radio	10^8	
Microwaves	$10^9\text{–}10^{11}$	
Infrared		$10^{-3}\text{–}10^{-6}$
Visible		$(8\text{–}4) \times 10^{-7}$
Ultraviolet		$4 \times 10^{-7}\text{–}10^{-9}$
X-rays		$10^{-8}\text{–}10^{-11}$
Gamma rays		$<10^{-11}$

Table 12.2 Frequency Range of Common Types of Electromagnetic Waves

Type of wave	Frequency (Hz)	Wavelength (m)
Power line	60	5×10^6
AM radio	$(0.5\text{–}1.5) \times 10^6$	600–200
FM radio	10^8	3
Microwaves	$10^9\text{–}10^{11}$	$0.3\text{–}3 \times 10^{-3}$
Infrared	$3 \times 10^{11}\text{–}3 \times 10^{14}$	$10^{-3}\text{–}10^{-6}$
Visible	$4 \times 10^{-14}\text{–}8 \times 10^{14}$	$(8\text{–}4) \times 10^{-7}$
Ultraviolet	$8 \times 10^{14}\text{–}3 \times 10^{17}$	$4 \times 10^{-7}\text{–}10^{-9}$
X-rays	$3 \times 10^{16}\text{–}3 \times 10^{19}$	$10^{-8}\text{–}10^{-11}$
Gamma rays	$>3 \times 10^{19}$	$<10^{-11}$

Solution

The relationship between frequency and wavelength is $c = \lambda f$. For power line frequencies of 60 Hz, the wavelength will be $\lambda = 3 \times 10^8/60 = 5 \times 10^6$ m. For am radio, the wavelength will vary between $\lambda = 3 \times 10^8/0.5 \times 10^6 = 600$ m and $\lambda = 3 \times 10^8/1.5 \times 10^6 = 200$ m. Repeating this calculation for all the given frequencies allows us to complete the table where wavelengths are missing. Where wavelengths are given, we calculate the frequencies using $f = 3 \times 10^8/\lambda$. For instance, in the case of infrared radiation, the frequency ranges from $3 \times 10^8/10^{-3} = 3 \times 10^{11}$ Hz to $3 \times 10^8/10^{-6} = 3 \times 10^{14}$ Hz. Table 12.1 then becomes as shown in Table 12.2.

The limits given in the Table 12.2 are only approximate, and the various types actually overlap considerably. For instance, microwaves and infrared radiation include the wavelengths around 10^{-3} m, and are identical waves irrespective of whether they are called microwaves or infrared. One usually distinguishes between them on the basis of how they were produced. If they were produced electronically, they are called microwaves. If they are produced from heat, they are called infrared. Similar distinctions are made at the boundaries of the different types of radiation. All of these waves travel with a speed of c, all are transverse, and all carry perpendicular electric and magnetic fields with them.

12.5 MATHEMATICAL DESCRIPTION OF ELECTROMAGNETIC WAVES

As in the case of a sound wave (and any other type of wave), the disturbance that is carried by the wave varies with both time and space. A plane wave travels in one dimension with its disturbance depending only on time and the position along the direction of travel. Suppose the wave is traveling in the x direction. At any point x and instant t every point in the plane parallel to the y–z plane at that x, has the same disturbance. As time changes the disturbance at all points in this plane change in lock step, i.e in phase.

The equation for the disturbance of a electromagnetic sinusoidal plane wave, traveling in the $+x$ direction, is given in terms of its disturbance (an electric field in the y direction) by

$$E = E_0 \cos 2\pi(ft - x/\lambda) = E_0 \cos(\omega t - kx), \tag{12.9}$$

where

$$\omega = 2\pi f \quad \text{and} \quad k = 2\pi/\lambda \tag{12.10}$$

Here ω is the angular frequency of the wave, and k is the "wavenumber" of the wave, and has units of m^{-1}. E_0 is the maximum value of the electric field, and is thus the amplitude of the wave. For a wave traveling in the $-x$ direction, the equation for the electric field is

$$E = E_0 \cos 2\pi(ft + x/\lambda) = E_0 \cos(\omega t + kx) \tag{12.11}$$

This equation will suffice for any single plane wave, since we can choose the direction of travel to be the x direction.

Problem 12.10. An electromagnetic plane wave is traveling in the x direction. The electric field is given by $E = 1.5 \cos(6 \times 10^4 t - 2 \times 10^{-4} x)$. Assume standard units.

(*a*) What is the amplitude of the wave?

(*b*) What is the frequency of the wave?

(*c*) What is the wavelength of the wave?

Solution

(*a*) The amplitude is the maximum electric field in the wave, which is given by the factor before the cosine function. Thus, the amplitude is $E_0 = 1.5$ V/m.

(*b*) We can see by comparing the general formula [Eq. (*12.9*)] to the specific equation given in this problem, that $\omega t = 6 \times 10^4 t$, or $\omega = 6 \times 10^4$ rad/s. Then $f = \omega/2\pi = 9.5 \times 10^3$ Hz.

(*c*) Again, comparing the general equation to the specific numbers in our equation, $kx = 2 \times 10^{-4}x$, or $k = 2 \times 10^{-4}\ \text{m}^{-1}$. Then $\lambda = 2\pi/k = 2\pi/2 \times 10^{-4}$, or $\lambda = 3.14 \times 10^4$ m. Note that this is consistent with the requirement that $f\lambda = c = 3 \times 10^8$ m/s.

Problem 12.11. An electromagnetic plane wave is traveling in the $-x$ direction. The electric field has an amplitude of 2 V/m, and a frequency of 1000 Hz. Write down an equation for the wave as a function of time and distance.

Solution

The general equation for an electromagnetic wave traveling in the $-x$ direction, is given by Eq. (*12.9*), $E = E_0 \cos 2\pi(ft + x/\lambda) = E_0 \cos(\omega t + kx)$. In our case, $E_0 = 2, f = 10^3$ Hz, and $\lambda = c/f = 3 \times 10^5$ m. Thus this wave is given by $E = 2 \cos [2\pi(1000t + x/3 \times 10^5)] = 2 \cos (6.28 \times 10^3 t + 2.09 \times 10^{-5}x)$.

Problem 12.12. An electromagnetic plane wave is traveling in the x direction. The electric field has an amplitude of 0.5 V/m, and a frequency of 2500 Hz. Write down an equation for the magnetic field component of the wave as a function of time and distance.

Solution

The general equation for the electric field of an electromagnetic wave traveling in the x direction, is given by Eq. (*12.9*), $E = E_0 \cos 2\pi(ft - x/\lambda) = E_0 \cos(\omega t - kx)$. For the magnetic field part of the wave, we will have the same general equation, except that the amplitude will be different and the direction of the magnetic field given by the formula, will be in the z direction rather than the y direction. Then, $B = B_0 \cos 2\pi(ft - x/\lambda) = B_0 \cos(\omega t - kx)$. In our case, $E_0 = 0.5$ V/m, $f = 2500$ Hz, $\lambda = c/f = 1.2 \times 10^5$ m and $B_0 = E_0/c = 1.67 \times 10^{-9}$ T. Thus the magnetic field of this wave is given by $B = 1.67 \times 10^{-9} \cos 2\pi(2500t - x/1.2 \times 10^5) = 1.67 \times 10^{-9} \cos(1.57 \times 10^4 t - 5.23 \times 10^{-5}x)$.

While plane waves are particularly simple to describe (they are essentially one-dimensional) other relatively simple sinusoidal waves are also worth noting. One such wave is a cylindrical wave. In this case, the wave travels radially away from a straight line with the same speed in all radial directions. Now the surfaces of constant phase in the electric (or magnetic) fields are concentric cylinders about the line. Thus if one point on a given cylindrical surface corresponds to maximum amplitude (a crest) of electric field, all other points on the surface are also crests of the electric field. If the line, which is the symmetry axis for the concentric cylinders is along the z axis, then the electric and magnetic fields will depend only on x and y. Actually the magnitude of the fields depends only on the radial distance, r, from the line. The direction of the fields does depend on where in the x–y plane one is, and the direction of propagation is along the radial direction, which is different for different x, y positions. The overall effect, however, is much like the expanding ripple in a pond, except that it is now a cylindrical surface expanding at speed c rather than a circle. Another simple sinusoidal wave, discussed briefly in the context of sound waves, is the spherical wave. In this case the source of the electromagnetic disturbance is a small region approximated by a point, and the disturbance expands out in a spherical shell. These concentric spherical surfaces correspond to constant phase in the electric and magnetic fields. The directions of the fields are always tangent to the spherical surfaces and the direction of propagation is

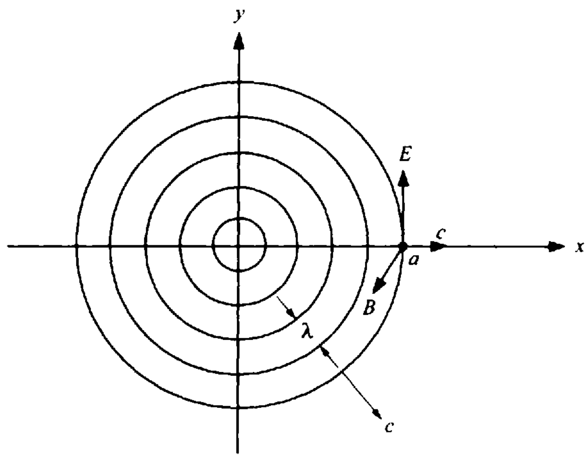

Fig. 12-7

always radially outward. While the direction of the electric field thus depends on the point in space (x, y, z), the magnitude depends only on the radial distance, r, to the point. The spherical shells of constant phase will be expanding in all directions with the speed c, and the radial distance between shells of adjacent crests will be the wavelength of the wave. We will still have the relationship between frequency, wavelength and speed given by $c = \lambda f$. As a given spherical shell expands out at speed c, its surface area will continually increase, and the energy of the wave (we will discuss this energy in the next section) which is uniformly spread over the surface will pass through larger and larger surfaces. The intensity of the wave, I, which is the energy per unit area perpendicular to the direction of propagation, therefore falls off. Since the area of a spherical surface is $4\pi r^2$, the intensity falls off as $1/r^2$. The amplitude of the wave, A, is related to the intensity by $I \propto A^2$, and therefore A falls off as $1/r$. In Fig. 12-7, we draw the intersection of constant phase spherical shells (corresponding to successive crests) centered on the origin with the x–y plane. The intersections correspond to concentric circles, separated by the wavelength λ. At every point of the spheres, the wave is moving radially outward with speed c. In the x–y intersection plane, the circles are moving outward with this same speed at each point. In particular, at point a, the wave is moving in the x direction with speed c. The direction of the electric field (the disturbance) is perpendicular to this direction, and in the y–z plane. Let us take it to be in the y direction. Then the magnetic field of the wave will be in the z direction. As one gets further and further away from the central point, the spheres have less and less curvature, and the wave begin to look more and more like plane waves (see e.g., Chap. 2 for an equivalent discussion for sound waves).

The spherical wave has a mathematical form similar to the plane wave. A major difference is that the electric field is now a function of distance, r, from a point, rather than of x, and that the amplitude gets smaller as r increases. The magnitude of E is constant over the surface of the spherical shell, just as it is constant over the surface of the plane for plane waves. However, the direction varies over the spherical surface so that it is always perpendicular to the radial direction. The formula for the magnitude of E is given by

$$E = (A/r)\cos 2\pi(ft - r/\lambda) = (A/r)\cos(\omega t - kr) \tag{12.12}$$

We will use this relationship later on when we discuss the energy and momentum carried by electromagnetic waves.

12.6 ENERGY AND MOMENTUM FLUX OF ELECTROMAGNETIC WAVES

We already showed previously that electric and magnetic fields contain energy. The energy density was shown to be $u_E = \varepsilon_0 E^2/2$ for electric fields and $u_B = B^2/2\mu_0$ for magnetic fields. The electromagnetic energy of an electromagnetic wave is just the sum of the energies of its electric and magnetic fields. The maximum energy is located at those points where the fields are at their maxima, which occurs at the crests of these waves. But these crests move with time at a speed of c, and therefore the energy is transported in the direction that the wave travels at this speed. An electromagnetic wave, therefore carries energy with it, just as do sound waves or other waves traveling through a medium, and in fact it also carries momentum.

To calculate the energy carried by an plane electromagnetic wave we would proceed in two steps. First we would calculate the total energy contained between successive crests of an electromagnetic wave. Then we would calculate the average energy that this wave transports per unit area and time as it travels with speed c in the x direction, which is defined as the intensity, I, of this wave.

The result is that:

$$I = c^2 \varepsilon_0 E_0 B_0/2 = E_0 B_0/2\mu_0 = c\varepsilon_0 E_0{}^2/2 \tag{12.13}$$

where we have used the fact that $E_0 = cB_0$. We see that the intensity of an electromagnetic wave is proportional to the amplitude squared, as was true for sound waves.

This quantity is often assigned a direction, **the direction of propagation**, and called the average **Poynting vector**. This vector is the average of the instantaneous Poynting vector, **S**, whose magnitude is EB/μ_0, and whose direction is perpendicular to **E** and **B**, and obeying the right-hand rule with finger curling from **E** to **B**. This is the direction of **c**, the propagation velocity. The Poynting vector, **S**, represents the instantaneous energy transported through unit area per unit time by the wave. It is actually more general than that, representing the energy transport even if the fields do not represent waves. The unit for intensity is $\text{J/m}^2 \cdot \text{s} = \text{W/m}^2$, since energy/s is power, or J/s = W.

Problem 12.13. An electromagnetic plane wave is traveling in the x direction. The formula for the wave is given by $E = (1.5 \text{ V/m}) \cos [(6 \times 10^4 \text{ s}^{-1})t - (2 \times 10^{-4} \text{ m}^{-1})x)]$. Calculate the intensity of this wave.

Solution

The intensity of a wave is $I = c\varepsilon_0 E_0{}^2/2$. For the above wave, $E_0 = 1.5$ V/m, and therefore the intensity is $I = 3 \times 10^8(8.85 \times 10^{-12})(1.5^2)/2 = 3.0 \times 10^{-3}\ \text{W/m}^2$.

Problem 12.14. An electromagnetic plane wave is traveling in the x direction. The intensity of this wave is $5 \times 10^{-3}\ \text{W/m}^2$. Calculate the maximum electric and magnetic fields of this wave.

Solution

The intensity of a wave is $I = c\varepsilon_0 E_0{}^2/2 = 5 \times 10^{-3}$. Thus $E_0 = [2(5 \times 10^{-3})/(3 \times 10^8)(8.85 \times 10^{-12})]^{1/2} = 1.94$ V/m. The maximum magnetic field, B_0 is $E_0/c = 6.5 \times 10^{-9}$ T.

Problem 12.15. An electromagnetic spherical wave is traveling outward from a point source. The intensity of this wave is $5 \times 10^{-3}\ \text{W/m}^2$, when the distance from the source is 2 m. Calculate the intensity of this wave when the distance is 5 m.

Solution

The intensity of a plane wave is $I = c\varepsilon_0 E_0{}^2/2$. For a spherical wave, Eq. (*12.12*), at any point in space the amplitude is $E_{max} = A/r$, instead of a constant. Therefore, the intensity of a spherical wave is given by $I = c\varepsilon_0(A/r)^2/2 = c\varepsilon_0 A^2/2r^2$. The intensity is therefore seen to decrease with distance from the source as $1/r^2$.

The intensity at $r = 2$ is given as 5×10^{-3}, so that $I_2 = 5 \times 10^{-3} = (c\varepsilon_0 A^2/2)(1/2)^2$. The intensity at $r = 5$ is $I_5 = (c\varepsilon_0 A^2/2)(1/5)^2$. Thus $I_5/I_2 = (2/5)^2 = 0.16$, or $I_5 = 0.16(5 \times 10^{-3}) = 8 \times 10^{-4}\ \text{W/m}^2$.

The $1/r^2$ dependence of the intensity could have been derived in a different manner. The energy per unit time, or power, from the source travels away uniformly in all directions. All the power emitted by the source flows through the surface of a sphere enclosing the source. For concentric spheres at radii r_1 and r_2, the power is evenly distributed over a surface area of πr_1^2 and πr_2^2, respectively. The intensity, which is the power per unit area, is therefore dependent on the distance as $1/r^2$, since we divide the power by the area to get the intensity.

Problem 12.16. An electromagnetic spherical wave is traveling outward from a point source. The intensity of this wave is 5×10^{-3} W/m^2, when the distance from the source is 2 m. Calculate the maximum electric field of this wave when the distance is 2 m and when the distance is 5 m.

Solution

For a spherical wave at any point in space the amplitude is $E_{max} = A/r$, the maximum electric field at that distance. Therefore, the intensity of a spherical wave is given by $I = c\varepsilon_0(E_{max})^2/2$. Since intensity decreases with distance from the source as $1/r^2$, (Problem 12.20), E_{max} decreases as $1/r$.

At $r = 2$, the intensity is given as $5 \times 10^{-3} = 3 \times 10^8(8.85 \times 10^{-12})E_{max}^2/2$, so $E_{max} = 1.94$ V/m, as obtained in Problem 12.19. To get E_{max} at $r = 5$, we use the fact that E_{max} depends on r as $1/r$, giving $E_5 = (\frac{2}{5})E_2 = 0.4(1.94) = 0.78$ V/m.

Whenever energy moves in a certain direction, there is also a certain amount of momentum in that direction. For instance, a particle with kinetic energy $(\frac{1}{2})mv^2$ has a momentum given by mv. One can show, using the equations of electromagnetic theory, that an electromagnetic wave also has momentum, but the calculation is not simple. We therefore present only the result of the calculation. We get that the average momentum density, which is just the average momentum per unit volume in the region where the plane wave exists is:

$$\text{momentum density} = S_{av}/c^2 = E_0 B_0/2c^2\mu_0 \tag{12.14}$$

with the direction of **S** in the direction of the momentum of the wave.
From Eq. (*12.13*) we see that the intensity I can be expressed as:

$$\text{Energy flux density} = I_{av} = S_{av} = E_0 B_0/2\mu_0 \tag{12.15}$$

and the energy density (from earlier work) is just

$$\text{energy density} = u = S_{av}/c = E_0 B_0/2\mu_0 c \tag{12.16}$$

Finally, the momentum that passes through an area A perpendicular to the direction of propagation in time Δt is given by:

$$\text{momentum flux density} = S_{av}/c = E_0 B_0/2c\mu_0 \tag{12.7}$$

These formulas give the average values for these quantities for a sinusoidal wave, where we have S is $E_0 B_0/2\mu_0$. If we replace $E_0 B_0/2$ by EB and S_{av} by S in Eqs. (*12.14*)–(*12.17*) we get the instantaneous values for the given quantities.

Problem 12.17. Sun light above the earth's atmosphere has an average intensity of approximately 1.4 kW/m^2. If this sunlight is absorbed by a solar panel with an area of 5 m^2, oriented perpendicular to the radiation direction, calculate (*a*) the energy absorbed per second; (*b*) the momentum absorbed per second; and (*c*) the force exerted on the solar panel by the sunlight.

Solution

(*a*) The intensity of the electromagnetic wave is $I = S_{av} = 1.4 \times 10^3$ W/m^2. The total energy absorbed, per second, by the panel is $IA = 1.4 \times 10^3(5) = 7 \times 10^3$ W.

(*b*) To get the momentum absorbed by this area per second, we multiply the momentum flux density by the area. Thus the momentum absorbed is $(S_{av}/c)A = 1.4 \times 10^3(5)/3 \times 10^8 = 2.33 \times 10^{-5}$ N.

(c) Whenever there is a change in momentum, there is a force causing this change in momentum. We know that for an object of mass m the force equals $ma = m\Delta v/\Delta t = \Delta p/\Delta t$. Thus, the force equals the rate of change of the momentum. If the radiation has a momentum flux of S/c, then the solar panel is absorbing SA/c units of momentum per second. This is the rate at which the momentum of the wave is decreased. The panel exerts the force that causes this change in momentum which equals $F = \Delta p/\Delta t = SA/c = 2.33 \times 10^{-5}$ N in the direction opposite to S. By **Newton's third law**, there is a reaction force of the wave on the panel of equal magnitude. In effect, the sunlight exerts a force on the panel.

This example shows how one can use sunlight in space to not only supply power but to exert a force on a spacecraft. The force that is exerted on the surface can best be characterized by the force exerted per unit area, or pressure, $P = F/A$. This "**radiation pressure**" equals, for a totally absorbing surface, $(SA/c)/A = S/c$ = momentum flux density. We will see in the supplementary problems that a totally reflecting surface is subject to twice this force (but it absorbs no energy).

For electromagnetic waves in ordinary matter, all the above equations still apply, provided that we use ε for ε_0, and μ for μ_0.

Problems for Review and Mind Stretching

Problem 12.18. A parallel plate capacitor is being charged at a rate of 5×10^{-3} C/s. Calculate the displacement current between the plates.

Solution

We know that the answer has to be 5×10^{-3} A, since the current in Ampere's law must be the same inside the plates as it is outside the plates. We will nevertheless perform the calculation to show that this is true. The electric field between the plates is V/d, where V is the potential difference across the plates, and d is the distance between the plates. But $V = Q/C$, and therefore $E = Q/Cd$. Now $C = \varepsilon_0 A/d$, giving $E = Qd/A\varepsilon_0 d = Q/A\varepsilon_0$. The displacement current is $\varepsilon_0 A\Delta E/\Delta t = \varepsilon_0 A(\Delta Q/\Delta t)/\varepsilon_0 A = \Delta Q/\Delta t = I = 5 \times 10^{-3}$ A.

Problem 12.19. A lightning bolt produces a flash of light and associated peal of thunder. The light is an electromagnetic radiation traveling at the speed of electromagnetic waves, while the thunder is a sound wave traveling with a speed of 345 m/s. If an observer hears the thunder 7.5 s after he sees the lightning, how far away did the lightning strike?

Solution

If the distance is called D, then the time it takes the lightning to reach the observer is $D/3.0 \times 10^8$, and the time for the thunder is $D/345$. The difference in time is 7.5 s, and equals $D[1/345 - 1/3.0 \times 10^8] = D/345$, since $3.0 \times 10^8 \gg 345$. Thus $7.5 = D/345$, $D = 2.6 \times 10^3$ m.

Note. The time for the light to travel this distance is $2.6 \times 10^3/3.0 \times 10^8 = 8.7 \times 10^{-6}$ s, confirming our assumption that the full 7.5 s represented the time for the sound wave to reach the observer.

Problem 12.20. How far does light travel in one year?

Solution

The distance is $ct = 3.0 \times 10^8 \times [365 \times 24 \times 60 \times 60] = (3.0 \times 10^8\ \text{m/s})(31.536 \times 10^6\ \text{s}) = 8.95 \times 10^{31}$ m $= 9.46 \times 10^{15}$ km. This distance is called a "**light-year**".

Problem 12.21. A powerful laser produces an electromagnetic plane wave. The power given to the wave is 10 MW, and the light beam is confined to an area of 2 mm^2. What is the intensity of the laser beam?

Solution

The intensity of any electromagnetic wave is the power passing unit area. Thus, the intensity of this beam is $(1.0 \times 10^7 \text{ W})/(2 \times 10^{-6} \text{ m}^2) = 5 \times 10^{12} \text{ W/m}^2$.

Problem 12.22. Sun light above the earth's atmosphere has an average intensity of approximately 1.4 kW/m^2.

(*a*) What is the maximum electric and magnetic field in this wave?

(*b*) What is the maximum force exerted by these fields on an electron moving with a velocity of 10^6 m/s in this sunlight?

Solution

(*a*) The intensity is given by $I = c\varepsilon_0 E_0{}^2/2 = 1.4 \times 10^3 \text{ W/m}^2$. Thus $E_0 = [2(1.4 \times 10^3)/(3 \times 10^8)(8.85 \times 10^{-12})]^{1/2} = 1.03 \times 10^3$ V/m. The magnetic field is $B_0 = E_0/c = 3.4 \times 10^{-6}$ T.

(*b*) The maximum force (magnitude) exerted by the electric field on the electron is $eE_0 = (1.6 \times 10^{-19} \text{ C})(1.03 \times 10^3 \text{ V/m}) = 1.6 \times 10^{-16}$ N. The maximum magnetic force is $evB_0 = 1.6 \times 10^{-19} (10^6)(3.4 \times 10^{-6}) = 5.4 \times 10^{-19} \text{ N} \ll$ electric force.

Note. For the magnetic force on the electron to be the same as the electric force the electron would have to be traveling at exactly the speed of light—which is not possible according to the theory of relativity.

Supplementary Problems

Problem 12.23. The displacement current through an area of 5×10^{-4} m^2 is 3 mA. What is the rate at which the electric field is changing in this region?

Ans. 6.78×10^{11} V/m · s

Problem 12.24. How long does it take for light to travel to (*a*) the moon; (*b*) the sun; and (*c*) a star at a distance of 3 light years?

Ans. (*a*) 1.27 s; (*b*) 497 s = 8.3 min; (*c*) 3 years

Problem 12.25. In air, light travels with a velocity that is 0.03% smaller than in vacuum. What is the difference in time that it takes for light to travel 10^3 m in air and in vacuum?

Ans. 1.00×10^{-9} s

Problem 12.26. Water has a dielectric constant of 1.77 at the frequencies of visible light, and essentially no magnetic properties. What is the velocity of light in water?

Ans. 2.25×10^8 m/s

Problem 12.27. An electromagnetic wave is traveling in the $+y$ direction with its electric field in the $+x$ direction. What is the direction of the magnetic field?

Ans. $-z$ direction

Problem 12.28. What is the frequency of an electromagnetic wave whose wavelength equals the diameter of the earth?

Ans. 23.4 Hz

Problem 12.29. An electromagnetic wave is given by $E = 20 \cos 2\pi(ft - 3.3 \times 10^{-7}x)$ in standard units.

(*a*) What is the amplitude of the wave?

(*b*) What is the wavelength of the wave?

(*c*) What is the frequency of the wave?

Ans. (*a*) 20 V/m; (*b*) 3.03×10^6 m; (*c*) 99 Hz

Problem 12.30. Sunlight near the surface of the earth has an intensity of 1.1×10^3 W/m^2. A lens, of diameter 6 cm, concentrates the sunlight it collects onto a circle of diameter of 1.5 mm, as in Fig. 12-8. What is the intensity of the light at this small circle?

Ans. 1.76×10^6 W/m^2

Problem 12.31. A 100 watt light bulb is 25% efficient in converting electrical energy into light. What is the intensity of the light from the bulb at a distance of 2 m?

Ans. 0.50 W/m^2

Problem 12.32. Sunlight has an intensity of 1.4×10^3 W/m^2. The light is totally reflected from a surface of area 6.5 m^2.

(*a*) What is the force on the surface?

(*b*) What is the radiation pressure on the surface?

Ans. (*a*) 6.07×10^{-5} N; (*b*) 9.33×10^{-6}

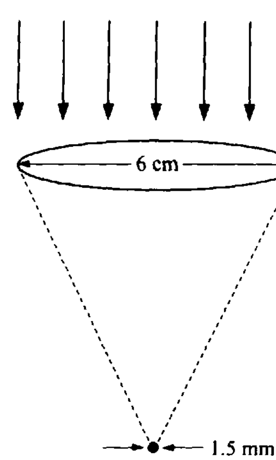

Fig. 12-8

Problem 12.33. An electromagnetic wave has a frequency of 3×10^6 Hz, and an intensity of 1.8 mW/m^2.

(*a*) What is the amplitude of the wave?

(*b*) What is the wavelength of the wave?

(*c*) What is the radiation pressure of the wave on a totally absorbing surface?

Ans. (*a*) 1.16 V/m; (*b*) 100 m; (*c*) 6×10^{-12} kg · m/s