Chapter 4

Electric Potential and Capacitance

4.1 POTENTIAL ENERGY AND POTENTIAL

In the previous chapter we learned about the force due to the electrical interaction and the electric field concept used to describe that force. The interaction is very similar to the interaction of masses with each other described by the gravitational interaction. Forces in general, as we learned in Chap. 6 of Beginning Physics I, Sec. 6.3, are able to do work, and the work that they do can be transformed into kinetic energy. For forces that are "conservative" the work done can be expressed in terms of a change in potential energy associated with those forces. In the case of the gravitational force due to the Earth, for example, the potential energy is given by U_p = mgh near the surface of the earth (where the force of gravity is a constant) and, more generally, $U_p = -GmM/r$ for greater distances r from the center of the earth. When some forces are conservative and others are not, the work–energy theorem can be expressed as total work (non-conservative) equals the total change in kinetic energy plus the total change in potential energy (due to all conservative forces). We now consider the electrical force. Is this force also conservative, and, if so, what is its potential energy?

Problem 4.1. By analogy to the force of gravitation (*a*) show that the electric force is conservative and (*b*) derive the formula for the potential energy of two charges, q and Q, separated by a distance r.

Solution

(*a*) The force of gravity is given in magnitude by $F_g = GmM/r^2$, and is a force of attraction along the line joining the masses. The electrical force between charges q and Q is given in magnitude by $F_e = kqQ/r^2$, and is a force along the line joining the charges. This force is attractive for charges of opposite sign and negative for charges of the same sign. When this force is attractive it is identical to the force of gravity if one interchanges charges for masses and the constant k for G. Therefore, it is clearly also conservative just as the force of gravity is conservative. If the force is between charges of the same sign, so that the force is repulsive, the work done by the force is the same as would be done by the same charges if they were of opposite sign, except that the work is the negative of that done by the attractive force. Since the attractive force is conservative, the work however depends only on the starting and ending points and not on what happened in between. This will also be true of the repulsive force which is therefore also conservative. Therefore the electric force is conservative, and work can be written in the form of a change in potential energy.

(*b*) By analogy with the force of gravity the potential energy can be written down immediately by substituting k for G and $-qQ$ for mM. We need the minus sign because for two positive charges the work is of the opposite sign to that two positive masses. The potential energy of two charges q and Q separated by a distance r is then given by:

$$U_p = kqQ/r = (1/4\pi\varepsilon_0)qQ/r \qquad (4.1)$$

A quick examination of signs shows that this equation works for arbitrary sign charges.

This formula can be used to calculate the potential energy for arbitrary sets of charges. This follows because energy is a scalar, and the total potential energy is determined by adding together, algebraically, the potential energy between pairs of charges.

We note that in Eq. (*4.1*) the zero of potential energy has been chosen when $r \to \infty$. If the charges are of the same sign then the potential energy increases as the charges approach each other. This follows because an external force must do positive work in forcing the charges closer together against their mutual repulsion. When such charges are left to themselves they try to move to regions of lower potential energy. This corresponds to the fact that the repulsive electrical force now does positive work by moving the charges further apart, thus causing a decrease in their potential energy. If the charges are of opposite sign then the potential energy becomes more negative (decreases) as the charges approach each other, and less negative (increases) as they are forced further apart. If left to themselves, these charges would move closer, seeking regions of lower potential energy.

If we fix the position of one charge, Q, and allow the second charge, q, to move, then the potential energy will vary with the position of the second particle. One could say that the system changes its potential energy and that this change in potential energy depends on the change in the position of the second charge. We could associate a specific potential energy with each point in space in a manner similar to associating an electric field to each point in space. From Eq. *(4.1)* we note that this potential energy is proportional to the moving charge. The potential energy per unit charge, U_p/q, then depends only on the position of the moving charge, as well as on the magnitude and sign of the stationary charge. Similarly, if one had many stationary charges, the potential energy of the entire system changes as the moving charge goes from one point to another, and is proportional to this moving charge. Again, the potential energy per charge depends only on the position of the moving charge and on the characteristics of the stationary charges. We can view this as a situation in which the stationary charges provide each point in space with a scalar value, called the potential, V, such that the potential energy of the system will equal qV if the moving charge is at that point in space. (We ignore here the potential energy between the fixed charges, which remains unchanged as the charge q moves.) The unit for potential V is the **volt** (V), which is the same as J/C. As the charge moves there will be a change in **potential energy**, ΔU_p which will equal q times the change in the potential at each point. In summary:

$$U_p = qV, \tag{4.2a}$$

and

$$\Delta U_p = q\Delta V \tag{4.2b}$$

The quantity ΔV is the "**potential difference**" between the two points, and depends on the stationary charges Q_i that produce this potential at all points in space. It is independent of the characteristics of the moving charge, q, whose potential energy changes. The potential is related to the potential energy in the same manner that the electric field is related to the electric force. Whenever an electric field is produced by some set of charges, Q_i, it acts as the source of the force distribution in space; it also can be thought of as the source of the potential distribution in space. If one places another charge, q, at some position in space, the electric field will exert a force of $\mathbf{F} = q\mathbf{E}$ on the charge, and the system will have a potential energy of $U_p = qV$, where $\mathbf{E}$ and V are the field and the potential at that point. The work done by the force $\mathbf{F} = q\mathbf{E}$ in moving the charge q from one location to another is just $-\Delta U_p = -q\Delta V$, from the usual relationship between work and potential energy. Clearly $\mathbf{E}$ and ΔV are related in exactly the same way that $\mathbf{F}$ and ΔU_p are related. This is discussed in greater detail in Sect. 4.3. One can change $\mathbf{E}$ and V by changing the source charges, Q_i and their position.

Problem 4.2. Two charges, $Q_1 = 3.3 \times 10^{-6}$ C and $Q_2 = -5.1 \times 10^{-6}$ C are located at the origin and at $x = 0.36$ cm, respectively. A third charge, $q = 9.3 \times 10^{-7}$ C, is moved from far away ($r = \infty$) to a point on the y axis, $y = 0.48$ cm.

(*a*) What is the potential energy between q and Q_1 at this point?

(*b*) What is the potential energy between q and Q_2 at this point?

(*c*) What is the change in potential energy of the system as one moves q from far away to this point?

(*d*) What is the potential difference between the point at ∞ and this point?

Solution

(*a*) The potential energy between any two charges is kqQ/r. Thus the potential energy between q and Q_1 is $U_p = (9.0 \times 10^9)(9.3 \times 10^{-7}\ \text{C})(3.3 \times 10^6\ \text{C})/0.48 \times 10^{-2}\ \text{m} = 5.75$ J.

(*b*) The distance between q and Q_2 is $(0.36^2 + 0.48^2)^{1/2}$ cm = 0.60 cm. Thus the potential energy between q and Q_2 is $U_p = (9.0 \times 10^9)(9.3 \times 10^{-7}\ \mathrm{C})(-6.3 \times 10^{-6}\ \mathrm{C})/0.60 \times 10^{-2}\ \mathrm{m} = -8.79$ J.

(*c*) When q is far away the potential energy between q and each of the charges Q is zero. There is potential energy of the system between Q_1 and Q_2, but that potential energy does not change as one moves q from point to point. As one moves the charge q to the final point the potential energy changes because of the interaction between q and the Q. The final potential energy is $U_p = 5.75\ \mathrm{J} - 8.79\ \mathrm{J} = -3.04$ J. Therefore $\Delta U_p = -3.04 - 0 = -3.04$ J.

(*d*) Since $\Delta V = \Delta U_p/q$, the potential difference is $\Delta V = -3.26 \times 10^6$ V.

4.2 POTENTIAL OF CHARGE DISTRIBUTIONS

The previous problem illustrated how to calculate the potential energy in the case of two fixed point charges and a moving charge, and then how to use that potential energy to obtain the potential. We can clearly use this procedure to calculate the potential produced by any number of point charges at all points in space. We can thus calculate the potential produced by a collection of particles or by a distribution of charge.

Problem 4.3. Calculate the potential produced by a point charge Q located at the origin at a point distant from the charge by r.

Solution

Our method is to calculate the potential energy, U_p, at the desired point if one places a "test charge" q at that point. Then the potential will equal U_p/q. Using Eq. (*4.1*), we get $U_p = kqQ/r$, and then:

$$V = kQ/r = (1/4\pi\varepsilon_0)Q/r \tag{4.3a}$$

This is the potential produced by a single charge Q at a point that is distant from the charge by r. If we have a collection of charges, Q_i, then the potential will equal:

$$V = k \sum Q_i/r_i = (1/4\pi\varepsilon_0) \sum Q_i/r_i \tag{4.3b}$$

Problem 4.4. A charge of 1.75×10^{-6} C is placed at the origin. Another charge of -8.6×10^{-7} C is placed at $x = 0.75$ m.

(*a*) What is the potential at a point halfway between the charges?

(*b*) What is the electric field at that point?

(*c*) If an electron is placed at that point, what force acts on it, and how much potential energy does it have?

Solution

(*a*) The potential equals $k \sum Q_i/r_i$. Thus $V = (9.0 \times 10^9)[(1.75 \times 10^{-6}\ \mathrm{C}/0.375\ \mathrm{m}) + (-8.6 \times 10^{-7}\ \mathrm{C}/0.375\ \mathrm{m})] = 2.14 \times 10^4$ V. Since V is a scalar we were able to add the values algebraically.

(*b*) To calculate the electric field we must calculate the magnitude and direction of the fields produced by each source and then add them vectorially. Thus $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. Now $|\mathbf{E}_1| = kQ_1/r^2 = (9.0 \times 10^9)(1.75 \times 10^{-6}\ \mathrm{C})/0.375^2 = 1.12 \times 10^5$ N/C. Since Q_1 is positive this field is directed along $+x$. Similarly, $|\mathbf{E}_2| = (9.0 \times 10^9)(8.6 \times 10^{-7}\ \mathrm{C})/0.375^2 = 5.50 \times 10^4$ N/C. Since Q_2 is negative, the field points toward Q_2 which is also in the $+x$ direction. Then the total field is 1.67×10^5 N/C in $+x$.

(*c*) An electron has a charge of -1.6×10^{-19} C. Therefore the force on it is $\mathbf{F} = q\mathbf{E} = (1.6 \times 10^{-19}\ \mathrm{C})(1.67 \times 10^5\ \mathrm{N/C}) = 2.67 \times 10^{-14}$ N. The direction is opposite to $\mathbf{E}$ since q is negative, so $\mathbf{F}$ is in $-x$. The potential energy is $qV = (-1.6 \times 10^{-19}\ \mathrm{C})(2.14 \times 10^4\ \mathrm{V}) = -3.42 \times 10^{-15}$ J.

Problem 4.5. Refer to the two fixed charges of Problem 4.4. At what two points on the x axis is the potential zero?

Solution

If the point of zero potential is between the charges, and the distance from the origin to the point is x, then the first charge is at a distance of x and the second charge is at a distance $(0.75 - x)$ from the point. The total field is $k[Q_1/x + Q_2/(0.75 - x)] = 0$. Q_1 is positive and Q_2 is negative. Substituting for the charges, we get: $(1.75 \times 10^{-6}/x) = 8.6 \times 10^{-7}/(0.75 - x)$. Then $(0.75 - x) = 0.49x$, $1.49x = 0.75$, $x = 0.50$ m. If the point of zero potential is not between the charges, and the distance from the origin to the point of zero potential is x, then the first charge is at a distance of x and the second charge is at a distance $(x - 0.75)$ from the point. (Recall that in Eq. (*4.3a*), r is always positive.) The total field is $k[Q_1/x + Q_2/(x - 0.75)] = 0$. Again, Q_1 is positive and Q_2 is negative. Substituting values for the charges, we get: $(1.75 \times 10^{-6}/x) = 8.6 \times 10^{-7}/(x - 0.75)$. Then $(x - 0.75) = 0.49x$, $0.51x = 0.75$, $x = 1.47$ m. A quick check for finite points on the negative x axis shows that the potential cannot vanish there. Of course, the potential also vanishes at $x \to \pm\infty$.

Problem 4.6. Four equal charges of 5.7×10^{-7} C are placed on the corners of a square whose side has a length of 0.77 m.

(*a*) What is the electric field at the center of the square?

(*b*) What is the electric potential at the center of the square?

(*c*) If one brought a charge of 6.8×10^{-7} C from rest at ∞ to the center of the square, what is the change in the potential energy of the system?

(*d*) How much work must be done by an outside force to bring in this charge?

Solution

(*a*) All the charges produce fields of the same magnitude at the center, since they have the same charge and are equidistant from the center. The charges at opposite corners produce fields that are in opposite directions, thus canceling each other. The total field at the center is therefore zero.

(*b*) The potential at the center is the sum of the contribution from each of the four charges. Each charge produces the same potential, kq/r, where r is the distance from the corner to the center. Thus $r = 0.77/\sqrt{2} = 0.544$ m. The total potential is therefore $V = 4(9.0 \times 10^9)(5.7 \times 10^{-7}\ \text{C})/0.544 = 3.77 \times 10^4$ V. We see that the potential can be non-zero even at a point where the electric field is zero.

(*c*) The change in the potential is the difference between the potential at the center of the square and the potential at ∞. Thus $\Delta V = 3.77 \times 10^4 - 0 = 3.77 \times 10^4$ V. The change in potential energy is $q\Delta V = (6.8 \times 10^{-7}\ \text{C})(3.77 \times 10^4\ \text{V}) = 0.026$ J. Thus the system gained 0.026 J of energy. (This makes sense since all the charges are positive so potential energy increases as the fifth charge is brought closer.)

(*d*) The work done by outside (non-conservative) forces equals the change in the total mechanical energy of the system. Since there is no change in kinetic energy, the outside work will equal the change in the potential energy, $W_{\text{outside}} = 0.026$ J.

Problem 4.7. A total charge of 5.4×10^{-6} C is uniformly distributed along a ring of radius 0.89 m.

(*a*) What is the potential at the center of the ring?

(*b*) What is the potential at a point on the axis of the ring at a distance of 0.98 m from the plane of the ring?

Solution

(*a*) All the charge is located at a distance of $r = 0.89$ m from the center of the ring. Each part of the charge therefore contributes the same scalar potential at the center, and the total potential is $kQ/r = (9.0 \times 10^9)(5.4 \times 10^{-6}\ \text{C})/(0.89\ \text{m}) = 5.46 \times 10^4$ V.

(*b*) Now all the charge is located at a distance of $(r^2 + x^2)^{1/2} = (0.89^2 + 0.98^2)^{1/2} = 1.32$ m, and the potential is $(9.0 \times 10^9)(5.4 \times 10^{-6}\ \text{C})/(1.32) = 3.68 \times 10^4$ V.

Note how easy it is to calculate the potential in Problem 4.7 in comparison with finding the electric field in a comparable problem in Chap. 3. This, of course, is a consequence of the potential being a scalar while the field is a vector.

4.3 THE ELECTRIC FIELD—POTENTIAL RELATIONSHIP

We know that the electric field is the force per charge and the potential is the potential energy per charge. The force and the potential energy are related by the work–energy theorem, and therefore the electric field and the potential must be related in the same manner. We would like to develop that relationship in more detail at this time. It is useful to do this by considering an opposing force to the electric force.

When an outside force (non-electric) **F**, is exerted on a charge in an electric field, and is adjusted to *always* be equal and opposite to the electric force, then the positive (negative) work done by that force in moving the charge from one location to another will equal the increase (decrease) in the electric potential energy of the charge. If no work is done by this outside force either because the force is zero (hence there is no electric field) or the force is perpendicular to the direction in which the charge moves, then there will not be any change in the electric potential energy of the charge. Therefore there is a change in potential energy (and a corresponding change in potential) only if there is a component of the electric field in the direction of motion. If one moves perpendicular to **E** [along $\Delta d_\perp$ in Fig. 4-1(*a*)], there is no change in V. If one moves in the direction of **E** [along $\Delta d_\parallel$ in Fig. 4-1(*a*)], then, for constant **E**, the change in potential energy is $|\mathbf{F}|d = -q|\mathbf{E}|\Delta d_\parallel$, and the change in potential will equal $\Delta V = -|\mathbf{E}|\Delta d$. If the field is at an angle of θ with the direction of motion (Δd in Fig. 4-1), then the change in potential will equal $\Delta V = -|\mathbf{E}|\Delta d \cos\theta$. If the field is not constant, then one must divide the path into small segments over which the field can be considered to be a constant and add the contribution from each segment. Thus, in general;

$$\Delta V = -\sum |\mathbf{E}| \cos\theta\, \Delta d, \tag{4.4}$$

where the sum is evaluated along the path of the particle [see Fig. 4-1(*b*)]. We have already learned that for a conservative force the result of this calculation depends only on the beginning and ending points, so we can choose any path between those points that we want in evaluating the sum. This relationship can be used to calculate ΔV between any two points if the field **E** is known along a path joining those points. Eq. (*4.4*) also shows that an equivalent unit for E is V/m.

Problem 4.8. Two parallel plates carry a surface charge density of $\pm\sigma$, respectively, and are separated by a small distance d. Assume that the size of the plates is always large compared with the distance to the plates.

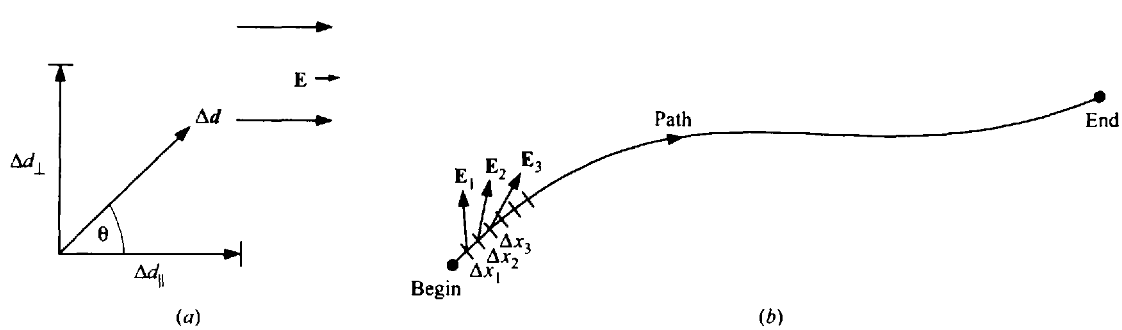

Fig. 4-1

(*a*) What is the electric field in the region between the plates?

(*b*) What is the potential difference between a point on one plate and a point on the other plate (e.g. points P_1 and P_2 in Fig. 4-2)?

(*c*) Which plate, the positive or the negative plate, is at the higher potential?

Solution

(*a*) We learned in the previous chapter that the field between the plates points from the positive plate to the negative plate, and has a constant magnitude of σ/ε_0.

(*b*) Since the field is constant and pointing along the direction perpendicular to the plates, we choose our path in two parts starting at the point P_1 as shown in Fig. 4-2. Along path 1 we move parallel to the field to the second plate, and along path 2 we move along the second plate, perpendicular to E, until the final point. Along path 2 there is no ΔV since we are moving perpendicular to **E**. Along path 1, $|\Delta V| = |E|d = \sigma d/\varepsilon_0$. Thus the potential difference is, in magnitude, equal to $\sigma d/\varepsilon_0$.

(*c*) Along path 1 the field is in the same direction as the displacement. Therefore, from Eq. (*4.4*), $\Delta V = V_2 - V_1 = -\sigma d/\varepsilon_0$, and the potential decreases as we move from the positive plate (P_1) to the negative plate (P_2), and the positive plate is at the higher potential, V_1. This illustrates the fact that the potential always decreases as we move along the direction in which the field points. Since the field points away from positive charge and towards negative charge, the potential decreases as we move away from positive or toward negative charge.

Problem 4.9. An isolated conducting sphere is charged with a total charge, Q, of 6.0×10^{-8} C, and has a radius of 1.35 m.

(*a*) What is the field inside the sphere, and what is the field outside the sphere?

(*b*) What is the potential at a distance r from the sphere, if r is outside the sphere?

(*c*) What is the potential at the surface of the sphere?

(*d*) What is the potential at a point r within the sphere?

(*e*) If instead of a conducting sphere we had a thin uniform spherical shell of charge, again with no other charges nearby, how would the answers to (a)–(d) change?

Solution

(*a*) We learned in Chap. 3 that the field inside a conductor is zero, and that the field outside an isolated conducting sphere, where the surface charge is uniformly distributed, is the same as if all the charge were concentrated at a point at the center of the sphere. Therefore the field is kQ/r^2 for $r > R$, and zero for $r < R$.

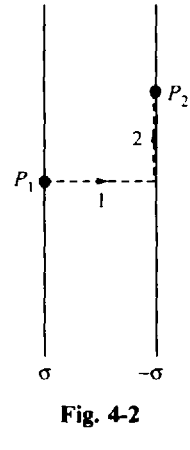

Fig. 4-2

(*b*) The field outside the sphere is identical to that of a point charge located at the center of the sphere. The sum to be evaluated [Eq. (*4.4*)] for the case of the sphere is therefore just the result for a point charge, as long as we remain outside the sphere. Therefore the difference in potential between a point at $r > R$ and a point at ∞ is $\Delta V = kQ/r$. Since the potential at ∞ is chosen to be zero, $V = kQ/r$.

(*c*) At the surface $r = R$. Thus $V_{\text{surface}} = (9.0 \times 10^9)(6.0 \times 10^{-8}\ \text{C})/1.35\ \text{m} = 400\ \text{V}$.

(*d*) The field inside the sphere is zero. Therefore if one moves from any point inside to any other point inside the sphere there will be no change in potential. The potential is the same everywhere within the sphere. At the surface the potential is 400 V, so the potential remains at 400 V for any other point $r < R$.

(*e*) By Gauss' law (choosing concentric spherical surfaces of radius $r < R$) since no charge is enclosed within the shell, the electric field will still be zero. The field outside could again be that of a point charge at the center so part (*a*) is unchanged. Similarly, the results of parts (*b*), (*c*) and (*d*) will be unchanged.

Problem 4.10. A charge Q_1 of 5.5×10^{-7} C is at the center of a conducting spherical shell that has an inner radius of 0.87 m and an outer radius of 0.97 m (see Fig. 4-3). The conducting sphere has a total charge of -2.3×10^{-7} C.

(*a*) How much charge Q_2 is there on the inner surface of the conducting sphere, and how much charge Q_3 is there on the outside surface?

(*b*) By adding the contributions from all charges, calculate the potential at a point at a distance of 1.05 m from the center.

(*c*) By adding the contributions from all charges, calculate the potential at a point at a distance of 0.95 m from the center.

(*d*) By adding the contributions from all charges, calculate the potential at a point at a distance of 0.45 m from the center.

Solution

(*a*) We know that in static equilibrium (no charges in motion) the electric field within the conducting shell is zero as it must be within any conductor. We draw a Gaussian surface at a radius within the conductor, and note that the flux through that surface is zero, since the field is zero. Therefore the total charge inside that surface must be zero. The only charges inside the surface are on the inner surface of the shell and at the center. Therefore the charge on the inner surface must be $Q_2 = -Q_1 = -5.5 \times 10^{-7}$

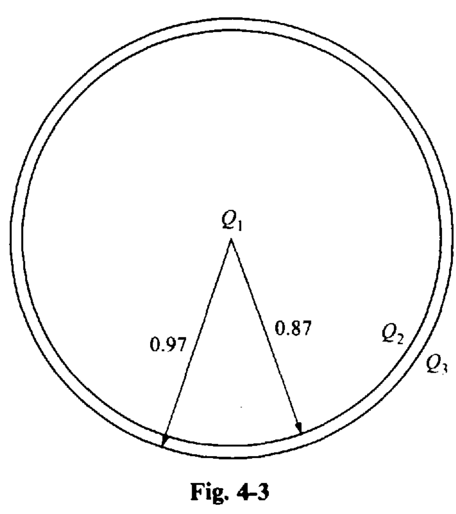

Fig. 4-3

C. The total charge on the sphere is given as -2.3×10^{-7} C which must equal $Q_2 + Q_3 = -2.3 \times 10^{-7} = Q_3 + (-5.5 \times 10^{-7})$, giving $Q_3 = 3.2 \times 10^{-7}$ C.

(*b*) We showed in the previous problems that the potential of a point charge is $V = kQ/r$. We also showed that the potential due to a uniform spherical surface charge distribution at a radius R is equal to $V_{\text{outside}} = kQ/r$ if $r > R$, and $V_{\text{inside}} = kQ/R$ if $r < R$. In our problem there are three charge distributions: a point charge at the center, a surface charge at $R = 0.87$ m and another surface charge at $R = 0.97$ m. If $r = 1.05$ m then we are seeking the potential outside each charge distribution. The total potential is then $V = V_1 + V_2 + V_3 = kQ_1/r + kQ_2/r + kQ_3/r = (9.0 \times 10^9)[(5.5 - 5.5 + 3.2) \times 10^{-7}\ \text{m}]/(1.05\ \text{m}) = 2.74 \times 10^3$ V.

(*c*) At $r = 0.95$ m, we are outside of charges Q_1 and Q_2, but within charge Q_3. Therefore $V_3 = kQ_3/R_3 = (9.0 \times 10^9)(3.2 \times 10^{-7}\ \text{C})/0.97\ \text{m} = 2.97 \times 10^3$ V. Furthermore, $V_1 + V_2 = k(Q_1 + Q_2)/r = (9.0 \times 10^9)(5.5 - 5.5) \times 10^{-7}/0.95 = 0$. Thus $V = 2.97 \times 10^3$ V.

Note. We could also have derived this result from the fact that E is zero within the conducting sphere, and therefore the potential within the sphere is the same as it is on the outer (or inner) surface. On the outer surface the potential, from part (*a*) is $k(3.2 \times 10^{-7})/0.97$, which is the same as we found.

(*d*) At $r = 0.45$ m, we are outside of the point charge but inside the two surface charges. The potential from the point charge is $kQ_1/r = (9.0 \times 10^9)(5.5 \times 10^{-7}\ \text{C})/0.45\ \text{m} = 1.1 \times 10^4$ V. The potential from the surface charges is $V_2 + V_3 = k(Q_2/R_2 + Q_3/R_3) = (9.0 \times 10^9)[(-5.5 \times 10^{-7}/0.87) + (3.2 \times 10^{-7}/0.97)] = -2.72 \times 10^3$ V. The total potential is then $1.1 \times 10^4 - 2.72 \times 10^3 = 8.29 \times 10^3$ V.

We have seen in the previous problems how to calculate the potential if the electric field is constant, or if the electric field is produced by a point charge, or if the electric field is produced by a spherical surface distribution. For other cases, one must use one of two methods to evaluate the potential difference between two points: (1) calculate the electric field everywhere along a path and then use the sum in Eq. (*4.4*) to calculate the difference in potential, or (2) use the charge distribution to calculate the potential at every point using Eq. (*4.3b*) and then calculate the difference between the potential at the points. We summarize some results from using such methods, together with the results we have already obtained.

For a point charge,

$$V = (1/4\pi\varepsilon_0)Q/r \tag{4.3a}$$

For a collection of charges,

$$V = (1/4\pi\varepsilon_0) \sum Q_i/r_i \tag{4.3b}$$

For a spherical surface charge at radius R;

$$V = (1/4\pi\varepsilon_0)Q/r \quad \text{for} \quad r > R \tag{4.5a}$$

and

$$V = (1/4\pi\varepsilon_0)Q/R \quad \text{for} \quad r < R \tag{4.5b}$$

For a long wire,

$$\Delta V = V_2 - V_1 = -(\lambda/2\pi\varepsilon_0) \ln (r_2/r_1) \tag{4.6}$$

for r_1 and r_2 any two perpendicular distances from the wire.

For a long cylinder of length L with symmetric surface charge on the cylindrical portion at radius R ($r, R \ll L$);

$$V = -(\lambda/2\pi\varepsilon_0) \ln (r/R') \quad \text{for} \quad r > R \tag{4.7a}$$

$$V = -(\lambda/2\pi\varepsilon_0) \ln (R/R') \quad \text{for} \quad r < R \tag{4.7b}$$

where R' is an arbitrary distance. It is often useful to set $V = 0$ at the radius of the cylinder, which is equivalent to setting $R' = R$.

For a large, uniformly charged infinitesimally thin plate of surface charge density σ,

$$\Delta V = V_2 - V_1 = -\sigma(|x_2| - |x_1|)/2\varepsilon_0 \tag{4.8}$$

where $|x_2|$ and $|x_1|$ are perpendicular distances on either side of the plate, and $|x_1|, |x_2| \ll L$, where L is the distance to the edge of the plate.

Problem 4.11. A coaxial cable (see Fig. 4-4) consists of a long, conducting wire, of radius R_1 with a linear charge density of λ, and a long conducting coaxial cylindrical shell, with an inner radius R_2 and an outer radius R_3, and with a symmetric linear charge density of $-\lambda$. We assume the length to be much greater than any of the radial distances of interest.

(*a*) What is the potential due to the cable at a point at a radial distance from the axis r, such that $r > R_3$?

(*b*) What is the potential at a point within the outer cylindrical shell, at $R_2 < r < R_3$?

(*c*) What is the potential at a point between the wire and the cylinder at $R_1 < r < R_2$?

(*d*) What is the potential at a point within the wire, at $r < R_1$?

Solution

(*a*) We use Eq. (*4.7a*) for each of the three surface charges since the point in question is outside both cylindrical distributions. Then $V = 0$, since the total enclosed linear charge density is $\lambda - \lambda = 0$.

(*b*) We note that the charge on the outer cylinder is all on the inner surface. This is because the field within the conductor is zero, and therefore, from **Gauss' law** the total charge within a Gaussian surface must be zero. Then the charge on the inner surface must cancel the charge on the wire, and equal $-\lambda$. Therefore the point within the cylinder is also outside all the charge distributions, and the result is the same as in (a), i.e. $V = 0$.

(*c*) In this case the point in question is outside of the wire but within the surface distribution on the outer cylinder. Using Eq. (*4.7a*) for the wire and Eq. (*4.7b*) for the cylinder we have for the potential: $V = V_1 + V_2 = (-\lambda/2\pi\varepsilon_0)\ \ln(r/R') - (-\lambda/2\pi\varepsilon_0)\ \ln(R_2/R') = (-\lambda/2\pi\varepsilon_0)\ \ln(r/R_2)$ (where we recall $\ln(A/B) = \ln A - \ln B$).

(*d*) Since we are now within the inner conducting cylinder where the field is zero, the potential must equal its value at the surface. Thus, $V = (-\lambda/2\pi\varepsilon_0) \ln(R_1/R_2)$.

Note. One could also get this result by adding the contributions of the two surface charge distributions. Then $V = V_1 + V_2 = (-\lambda/2\pi\varepsilon_0)\ \ln(R_1/R') - (-\pi/2\pi\varepsilon_0)\ \ln(R_2/R') = (-\lambda/2\pi\varepsilon_0)\ \ln(R_1/R_2)$.

Problem 4.12. Two large thin parallel plates are a distance D apart, and have surface charge densities

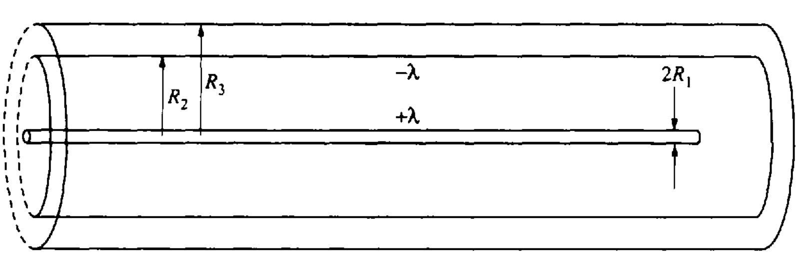

Fig. 4-4

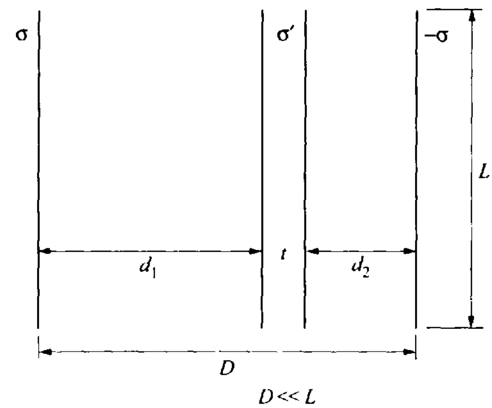

Fig. 4-5

of $\pm\sigma$, as in Fig. 4-5. A large conducting plate, of thickness t, is placed with one side at a distance of d_1 from the positive plate, as in the figure. The conducting plate has a charge density of σ'.

(*a*) What is the surface charge distribution on the two sides of the conducting plate?

(*b*) What is the difference in potential between the positive plate and the conducting plate?

(*c*) What is the difference in potential between the positive and the negative plates?

Solution

(*a*) The field within the conducting plate must be zero, as it is within any conductor. Each charge distribution produces a field of $\sigma/2\varepsilon_0$ pointing away from positive and toward negative charge. The field within the conductor has four contributions: (1) from the positive plate with charge distribution σ, (2) from the negative plate with charge distribution $-\sigma$, (3) from the side of the conducting plate near the positive charge with a charge distribution labeled σ_1, and (4) from the other side of the conducting plate with a charge distribution $\sigma_2 = (\sigma' - \sigma_1)$. The fields produced are: $E = E_1 + E_2 + E_3 + E_4 = (1/2\varepsilon_0)[\sigma + \sigma + \sigma_1 - (\sigma' - \sigma_1)] = (1/2\varepsilon_0)(2\sigma + 2\sigma_1 - \sigma') = 0$. Thus, $\sigma_1 = (\sigma'/2) - \sigma$. On the other side of the plate the charge distribution is then $\sigma_2 = (\sigma'/2) + \sigma$. (As a check we add $\sigma_1 + \sigma_2$ to get σ'.)

(*b*) To obtain the difference of potential between two points we calculate the field in the region between the points and, for a constant field perpendicular to the plates use the fact that $\Delta V = -E\Delta x$, where ΔV is the final-minus-initial potential as we move through Δx. In the region between the positive plate and the conducting plate, the field is $E = [\sigma - (\sigma'/2)]/\varepsilon_0$ to the right. We get this result either by adding the field from all four distributions or by using Gauss' law. By adding the contributions we get $E = (1/2\varepsilon_0)[\sigma - (\sigma'/2 - \sigma) - (\sigma'/2 + \sigma) - (-\sigma)] = [\sigma - (\sigma'/2)]/\varepsilon_0$. This field is to the right if the number is positive. Then the difference of potential between the positive plate and the conducting plate is given by $\Delta V = V_c - V_+ = -[\sigma - (\sigma'/2)]d_1/\varepsilon_0$, or $V_+ - V_c = [\sigma - (\sigma'/2)]/\varepsilon_0$.

(*c*) Using the same procedure we obtain the field between the conducting plate and the negative plate to be $E = [\sigma + (\sigma'/2)]/\varepsilon_0$. Then the difference of potential between the conducting plate and the negative plate is given by $\Delta V = V_- - V_c = -[\sigma + (\sigma'/2)]d_2/\varepsilon_0$. The difference of potential between the positive and the negative plates is therefore: $V_+ - V_- = (V_+ - V_c) + (V_c - V_-) = [\sigma - (\sigma'/2)]d_1/\varepsilon_0 + [\sigma + (\sigma'/2)]d_2/\varepsilon_0 = (1/\varepsilon_0)[\sigma(d_1 + d_2) + (\sigma'/2)(d_2 - d_1)]$.

4.4 EQUIPOTENTIALS

In our discussion so far we have learned how to use information about the electric field to obtain the potential difference between two points. We now shift our attention to the reverse process, obtaining the electric field from a knowledge of the potential. At every point there is an electric field pointing in some direction. If we move to a different point along that direction, then the potential will change.

However, if we move to a different point perpendicular to that direction, the potential will not change. Thus, for example, for the uniform field between large parallel plates, for every plane perpendicular to **E**, the potential remains the same at every point in the plane. Even for non-uniform fields, if we continue moving from point to point, always in a direction perpendicular to the electric field at that point, we will sweep out a surface with all points on that surface at the same potential. This surface is called the "**equipotential surface**". This idea can be used to obtain the direction of the electric field at any point if we know the potential everywhere in the region. We do this by sweeping out the various equipotential surfaces, and noting that the electric field lines are perpendicular to those surfaces. Once we have the direction of the electric field we can easily obtain its magnitude. We move a distance Δd in the direction of the electric field, between nearby equipotential surfaces and note the difference in potential. We know that along the direction of the electric field $\Delta V = -E\Delta d$, giving $E = -\Delta V/\Delta d$. The minus sign means that E is positive in the direction that ΔV is negative, i.e. **E** points from high to low potential. Thus, a knowledge of how V varies in a region around a point allows us to obtain the magnitude and the direction of the electric field at that point.

Problem 4.13. The potential produced by a point charge is $V = kQ/r$. Use this information to: (*a*) determine the shape of the equipotential surfaces, (*b*) determine the direction of the electric field at any point and (*c*) determine the actual value of the electric field at any point.

Solution

(*a*) The potential at a point at a distance r from the charge is given as $V = kQ/r$. All other points at the same distance r from the charge have the same potential. Therefore the equipotential surface consists of all points equidistant from the source at a distance r. This is the surface of a sphere of radius r. The equipotential surfaces are therefore concentric spherical surfaces.

(*b*) The direction of the electric field is perpendicular to the equipotential surfaces. That direction, for spheres, is in the direction of the radius. Thus the electric field must point along a radius. We know that it points from high to low potential. If Q is positive, then the potential decreases as r increases. Therefore the field points in the direction away from the charge, as we expected. For a negative charge the potential becomes less negative as r increases, which means that V increases as r increases. Then E points toward smaller r, or toward the center.

(*c*) The magnitude and direction of E along a radius is given by $|E| = \Delta V/\Delta d$, if Δd is along the direction of the field. Here $\Delta d = \Delta r$. If we move along a radius from r_1 to r_2, the difference in potential is $\Delta V = V_2 - V_1 = kQ(1/r_2 - 1/r_1) = kQ(r_1 - r_2)/r_1 r_2$. For very small $\Delta r = r_2 - r_1$ we can set $r_1 = r_2 = r$ in the denominator to get $\Delta V = -kQ\Delta r/r^2$. Then $E = -\Delta V/\Delta r = kQ\Delta r/r^2\Delta r = kQ/r^2$, as expected.

Problem 4.14. Two large parallel plates carry charge distributions of $\pm\sigma$. The positive plate is at $x = 0$, and the negative plate is at $x = d$, where x is measured perpendicular to the plates. The potential at any point can be shown to be given by $V = V_0(1 - x/d)$ when $0 < x < d$, i.e. between the plates, and where V_0 and 0 are the potentials at the positive and negative plates, respectively.

(*a*) What are the equipotential surfaces?

(*b*) What is the direction of the electric field at a point located at a distance x from the positive plate?

(*c*) What is the magnitude of the electric field at this point?

Solution

(*a*) The potential at a point at a distance x from the positive plate is given as $V = V_0(1 - x/d)$. All other points at the same distance x from the plate have the same potential. Therefore the equipotential surface consists of all points equidistant from the plate at a distance x. This surface is a plane parallel to the plates. The equipotential surfaces are therefore planes parallel to the plates.

(*b*) The direction of the electric field is perpendicular to the equipotential surfaces. That direction, for a plane parallel to the y-z plane, is in the direction of x. Thus the electric field must point along x. We know that it points from high to low potential. The potential decreases from V_0 to zero as one increases x from zero to d. Therefore the field is in the $+x$ direction.

(c) The magnitude of E is given by $|E| = \Delta V/\Delta x$, if Δx is along the direction of the field. If we move along the field from x_1 to x_2, the difference in potential is $\Delta V = V_2 - V_1 = V_0[(1 - x_2/d) - (1 - x_1/d)] = V_0(x_1 - x_2)/d = -V_0\,\Delta x/d$. Then $|E| = V_0\,\Delta x/d\Delta x = V_0/d$, as expected.

Problem 4.15. The electric field lines for a particular situation are shown in Fig. 4-6(*a*). Along the curved field line *OACD* the electric potential decreases linearly by 4.0 V every 3.0 m. At point *A* the potential, V_A, is 40 V.

(*a*) On the figure, draw the direction of the electric field at *A*.

(*b*) Calculate the magnitude of the electric field at *A*.

(*c*) Calculate the potential, V_C, at point *C*, which is 3.0 m from *A*.

(*d*) Calculate the potential, V_B, at point *B* which is 0.010 m along a line perpendicular to the field line through *A*.

Solution

(*a*) The field is tangent to the electric field line at any point. It points from high to low potential. Since the potential is decreasing as one moves along the line toward *C*, the field points in that direction. The direction is shown in Fig. 4-6(*b*).

(*b*) The magnitude of the field is equal to $\Delta V/\Delta x$ if one moves along the direction of E. When moving from *A* to *C* one is indeed moving in the direction of *E*, and $\Delta V/\Delta x = 4.0\ \text{V}/3.0\ \text{m} = |E| = 1.33\ \text{V/m}$. Ordinarily this would be the average magnitude of *E* over the 3.0 m distance, but because the potential decreases linearly it is the actual magnitude at any point along the line.

(*c*) We can obtain V_C from $\Delta V = V_C - V_A = -E\Delta x = -(1.33\ \text{V/m})(3.0\ \text{m}) = -4.0\ \text{V}$. Then $V_C = 40 - 4.0 = 36\ \text{V}$.

(*d*) Point *B* is along a direction perpendicular to the electric field. Therefore the potential does not change as one moves from *A* to *B*. Thus $V_B = V_A = 40\ \text{V}$.

The result that we have obtained for calculating the electric field from a knowledge of the potential everywhere can be written in a different form. If one moves a small distance Δx in the x direction from a given point, and the electric field makes an angle θ with the x axis at that point, then the change in potential in that direction, $\Delta V_x = -E\cos\theta\,\Delta x = -E_x\Delta x$. Thus $E_x = -\Delta V_x/\Delta x$, where ΔV_x is the

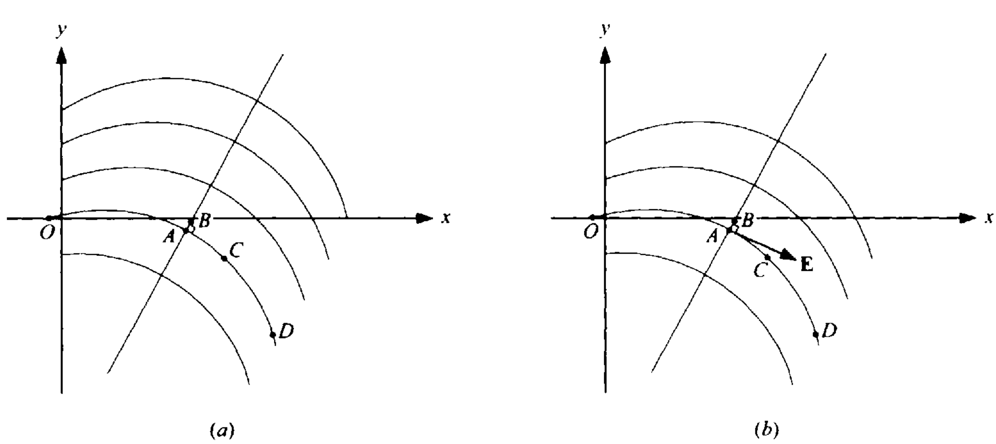

Fig. 4-6

change in V as one moves in the x direction. Similarly, $E_y = -\Delta V_y/\Delta y$, and $E_z = -\Delta V_z/\Delta z$. If we have the three components of the vector **E**, then we have all the information needed to characterize **E** at that point. The vector, whose components are determined by calculating the rate of change of V in each direction $(\Delta V_x/\Delta x, \Delta V_y/\Delta y, \Delta V_z/\Delta z)$, is called, in mathematical terminology, the **gradient** of V, and written as ∇V. Then our expression relating the electric field to the potential at every point in space can formally be expressed as $\mathbf{E} = -\nabla V$. As you may have guessed this is a calculus relationship and allows one to carry out sophisticated analyses beyond the scope of this book.

Problem 4.16. Fig. 4-7 shows the value of the electric potential at various points in the x–y plane. The potential at the origin is 75 V. At points along the x and y axes, at a distance of 0.65 m from the origin, the potentials are as shown.

(*a*) Calculate the x and y components of the electric field at the origin. Assume the potential varies linearly with distance in both the x and y directions.

(*b*) What is the magnitude and direction of the electric field at the origin?

(*c*) What can one say about the electric field at other points near the origin?

Solution

(*a*) To get E_x we must calculate $E_x = -\Delta V_x/\Delta x = -(65 - 75)\text{V}/0.65\text{ m} = 15.4\text{ V/m}$. Similarly, $E_y = -\Delta V_y/\Delta y = -(80 - 75)\text{V}/0.65\text{ m} = -7.7\text{ V/m}$. Thus the field has components in $+x$ and in $-y$ of 15.4 V/m and 7.7 V/m, respectively.

(*b*) $E = (E_x{}^2 + E_y{}^2)^{1/2} = 17.2\text{ V/m}$. If θ is the angle of E below the positive x axis, we have $\tan\theta = |E_y/E_x| = 0.50 \rightarrow \theta = 26.6°$.

(*c*) Since the potential varies linearly in the region from -0.65 m to $+0.65$ m in both the x and y directions, both E_x and E_y will be constant in that region. Thus **E** will be uniform for all points near the origin.

Problem 4.17.

(*a*) Show that the surface of a conductor (in static equilibrium) is always an equipotential surface irrespective of the charge on the surface or of nearby charges.

(*b*) Show that a hollow region inside a conductor that has no charges in it has no electric field in it as well.

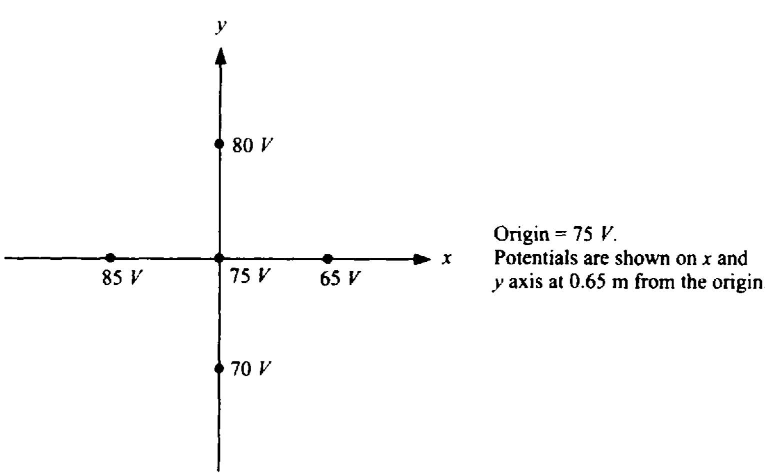

Fig. 4-7

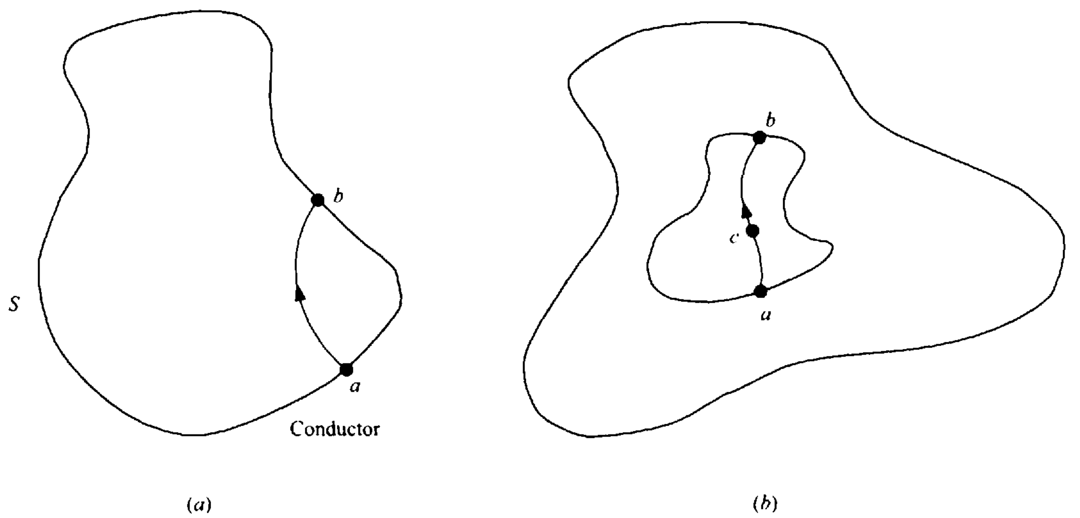

Fig. 4-8

Solution

(*a*) Consider the conductor shown in Fig. 4-8 with surface *S*, and consider two points on the surface, *a* and *b*. We can use Eq. (*4.4*) along any path leading from *a* to *b* to obtain $\Delta V = V_b - V_a$, including the path shown through the conductor. For the path chosen, which is wholly in the conductor, E is zero everywhere along the path. Therefore, $\Delta V = 0 \rightarrow V_b = V_a$. Since this is true for all points *a* and *b* on the surface, the surface must be an equipotential. (Indeed, the whole conductor is an equipotential, by the same argument.)

(*b*) Consider the hollow in the conductor shown in Fig. 4-8(*b*). Suppose there were an electric field at any point *c* in the hollow. If we trace the electric field line through point *c* it would have to start at some point *a* on the inner surface and end at some other point *b*. This is because the electric field lines always start and end on charges or go off to infinity. Since the electric field points in the same direction everywhere on the field line from *a* to *b*, applying Eq. (*4.4*) to the path along the field line, $\cos\theta$ is always equal to one and the sum must be a positive (non-zero) value. Therefore, $V_b - V_a \neq 0$ and the surface cannot be an equipotential. Since we have just shown in part (*a*) that it must be an equipotential, our hypothesis that an electric field existed at point *c* cannot be true. Since point *c* was chosen arbitrarily, we must have $\mathbf{E} = 0$ at all points in the hollow. (This implies that the hollow is also an equipotential region with the same value as the conductor.) This result is no longer true if a charge were placed in the hollow region.

4.5 ENERGY CONSERVATION

The potential energy associated with the electrical force can be used in the same manner as any other potential energy. We note that the potential energy of any charge is given by qV, and the change in potential energy that is used in most energy related problems is $\Delta U_p = q\Delta V$. A positive charge gains energy as it moves to a region of higher potential (ΔV positive) and, unless restricted by other forces, will tend to move to regions of lower potential. A negative charge, such as an electron, will lose energy as it moves to a higher potential (q negative and ΔV positive), and therefore tends to move to a region of higher potential. When an electron moves through a difference of potential of one volt it gains or loses $e(1) = 1.6 \times 10^{-19}$ J of energy. This amount of energy is called an **electron-volt**, or eV. If the electron moves through a difference of potential of x volts, the electron gains or loses x electron-volts of energy. This is a very convenient unit of energy to use whenever one discusses the motion of an electron, or other particle with a similar charge, since the energy the particle gains (loses) in eV is numerically equal to the difference of potential in volts through which it moves.

Problem 4.18. An electron moves from the positive to the negative terminal of a 9 V battery. How much potential energy did it gain or lose? Did it gain or did it lose potential energy?

Solution

The change in potential energy was 9 eV, since the electron moved through a difference of potential of 9 volts. This corresponds to (9 eV)(1.6×10^{-19} J/eV) = 1.44×10^{-18} J. Since the charge on the electron is negative, and the change in potential was also negative, the electron gained potential energy. This is in accordance with our discussion that negative charges tend to move to higher potentials in order to lose potential energy, and they gain potential energy in moving to lower potentials.

Problem 4.19. We want to produce protons with a kinetic energy of 4.3×10^{-15} J. Through what difference of potential should we accelerate them in order to obtain that kinetic energy, assuming that they start from rest and that there are no other forces present?

Solution

Since only the electric force is present, and the electric force is conservative, we can use conservation of energy in this problem. If we start with a stationary proton, then the proton has no initial kinetic energy. The increase in kinetic energy must equal the decrease in potential energy. Thus the positively charged proton must move through a difference in potential that will result in the loss of 4.3×10^{-15} J. This means that it must move through ΔV such that $q\Delta V = -4.3 \times 10^{-15}$ J, or $\Delta V = (-4.3 \times 10^{-15}\ \text{J})/1.6 \times 10^{-19}$ C $= -2.69 \times 10^4$ V. Alternatively, we could have converted 4.3×10^{-15} J into eV by dividing by 1.6×10^{-19} J/eV, obtaining 2.69×10^4 eV. Then we can say that a proton must have fallen through a decrease of 2.69×10^4 V to lose that amount of potential energy.

Problem 4.20. A proton is moving directly toward a fixed nucleus containing 23 protons. The speed of the proton when it is at a distance of 5.8×10^{-9} m from the nucleus is 2.4×10^6 m/s. The proton has a charge of 1.6×10^{-19} C and a mass of 1.67×10^{-27} kg.

(*a*) What was its kinetic and potential energy at this initial distance?

(*b*) At what distance from the nucleus does the proton stop, i.e. what is the distance of nearest approach? (Assume the nucleus remains stationary.)

Solution

(*a*) The kinetic energy of the proton is $(1/2)m_p v^2 = (0.5)(1.67 \times 10^{-27}\ \text{kg})(2.4 \times 10^6\ \text{m/s})^2 = 4.81 \times 10^{-15}$ J. The potential energy is $U_p = kqQ/r = (9.0 \times 10^9)(1.6 \times 10^{-19}\ \text{C})(23 \times 1.6 \times 10^{-19}\ \text{C})/(5.8 \times 10^{-9}\ \text{m})$ $= 9.14 \times 10^{-19}$ J. The total energy is therefore nearly all kinetic energy and equals 4.81×10^{-15} J.

(*b*) By conservation of energy, the total energy must be the same as the proton moves toward the nucleus. At the point of nearest approach, the kinetic energy is zero, since $v = 0$. Therefore, the potential energy must equal the original energy. Thus, $kqQ/r = 4.81 \times 10^{-15}\ \text{J} = (9.0 \times 10^9)(1.6 \times 10^{-19}\ \text{C})$ $(23 \times 1.6 \times 10^{-19}\ \text{C})/r = 5.30 \times 10^{-27}/r$. Then $r = 1.10 \times 10^{-12}$ m.

Problem 4.21. Four charged particles are placed at the corners of a square of side 0.39 m. The particles have charges of 2.3 μC, -5.6 μC, 7.9 μC and -1.3 μC as in Fig. 4-9.

(*a*) How much work was done by outside forces to place those particles in their positions if they were originally very far away?

(*b*) If an electron starts with no velocity very far away, what velocity does it have when it reaches the center of the square? ($m_e = 9.1 \times 10^{-31}$ kg)

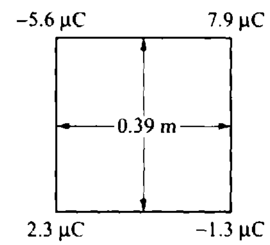

Fig. 4-9

Solution

(*a*) We will assemble the particles one at a time. To place the first particle (2.3 μC) in place requires no work ($W_1 = 0$) since there are no forces present as yet. To place the next particle (-5.6 μC) in place the outside work W_2 must be equal to the change in potential energy. This equals $W_2 = kQ_1Q_2/r_{12} = (9.0 \times 10^9)(2.3 \times 10^{-6}\text{ C})(-5.6 \times 10^{-6}\text{ C})/0.39\text{ m} = -0.30$ J. To place the next particle we must again supply the added potential energy. This additional potential energy is due to the interaction with both of the particles already in place. Thus $W_3 = kQ_3(Q_1/r_{13} + Q_2/r_{23}) = (9.0 \times 10^9)(7.9 \times 10^{-6})[(2.3 \times 10^{-6}/0.39\sqrt{2}) + (-5.6 \times 10^{-6}/0.39)] = -0.72$ J. Similarly, to add the fourth particle requires work of $W_4 = kQ_4(Q_1/r_{14} + Q_2/r_{24} + Q_3/r_{34}) = (9.0 \times 10^9)(-1.3 \times 10^{-6})[(2.3 \times 10^{-6}/0.39) + (-5.6 \times 10^{-6}/0.39\sqrt{2}) + (7.9 \times 10^{-6}/0.39)] = -0.19$ J. The total work is therefore $W_{\text{total}} = W_1 + W_2 + W_3 + W_4 = -0.30 - 0.72 - 0.19 = -1.21$ J.

(*b*) With all the four particles in place, the potential at the center is $V = V_1 + V_2 + V_3 + V_4 = k(Q_1 + Q_2 + Q_3 + Q_4)/r = (9.0 \times 10^9)(2.3 - 5.6 + 7.9 - 1.3) \times 10^{-6}/0.195\sqrt{2} = 1.08 \times 10^5$ V. At a large distance, the potential is zero. Therefore the electron loses potential energy equal to 1.08×10^5 eV. This is converted into kinetic energy. Then, $(1/2)mv^2 = (1.08 \times 10^5\text{ eV})(1.6 \times 10^{-19}\text{ J/eV}) = 1.73 \times 10^{-14}$ J. The mass of an electron is 9.1×10^{-31} kg, so $v^2 = 2(1.73 \times 10^{-14})/9.1 \times 10^{-31} = 3.80 \times 10^{16}$, and $v = 1.9 \times 10^8$ m/s.

Problem 4.22. Two large, thin parallel plates, of length L, are perpendicular to the x axis and carry charge distributions of $\pm\sigma$ (as in Fig. 4-10). The positive plate is at $x = 0$, and the negative plate is at $x = d$. The potential at any point is given as $V = V_0(1 - x/d)$ for $0 < x < d$, i.e. between the plates. An electron starts at the bottom, halfway between the plates, with an upward speed of v_0. The electron just passes the end of the plate at the top. Assume that the field is uniform throughout the region between the plates, and the potential is as given above. Give your answers in terms of L, d, v_0, σ and e (where e, as always, is the magnitude of the electron charge).

(*a*) How much kinetic energy, ΔK, did the electron gain until it leaves the region between the plates?

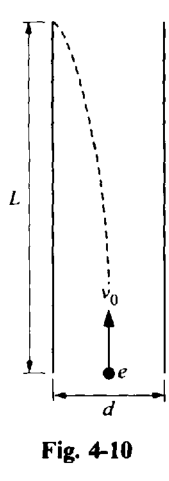

Fig. 4-10

(*b*) What is the x component of the velocity of the electron?

(*c*) How much time does it take for the electron to move through the plates?

Solution

(*a*) We will use conservation of energy to solve this part of the problem. The gain in kinetic energy ΔK must equal the loss in potential energy. This loss is equal to $e\Delta V = e(V_f - V_i) = e(V_0 - V_0 d/2) = eV_0 d/2$. Thus the gain in kinetic energy is $eV_0 d/2$. Recalling that the potential difference across the plates is just $(V_0 - 0) = Ed = \sigma d/\varepsilon_0$, we have finally $\Delta K = e\sigma d^2/2\varepsilon_0$.

(*b*) The gain in kinetic energy is $K_f - K_i = (\frac{1}{2})m(v_f^2 - v_i^2) = (\frac{1}{2})m(v_{fx}^2 + v_{fy}^2 - v_{iy}^2)$ where we recall $v_{ix} = 0$. Now, $v_y = v_0$ does not change, so $\Delta K = (\frac{1}{2})mv_{fx}^2$ and using our results in (*a*) we get: $v_{fx} = [(e/m)\sigma d^2/\varepsilon_0]^{1/2}$.

(*c*) Since v_y does not change, the time to move a distance of L in y is $t = L/v_0$.

Note. If we wanted we could solve for V_0, since we must also have $v_{fx} = at$ where acceleration $a = |(e/m)E| = (e/m)\sigma/\varepsilon_0$, and we can solve for t and insert in $t = L/v_0$.

4.6 CAPACITANCE

We have seen that positive work is required by an outside force to separate opposite charges that were initially together. For instance, we may have two metal surfaces which were initially uncharged, and then remove negative charge from one surface and place this charge on the other surface. The first surface that lost negative charge becomes positively charged, and the other surface gains the same negative charge. The more charge that we transfer the harder it becomes to transfer the next unit of charge because of the Coulomb forces between the charges, and the more work we have to do to transfer additional charge. This work is manifested in the resultant potential energy of the final distribution of charge.

When a given distribution of charge is reached, we wish to be able to calculate the potential everywhere in space. This will allow us to determine the energy necessary to bring another charge from one location to another. We know that each conductor surface will be an equipotential surface once charges have reached their equilibrium positions. Therefore each surface has its own potential and potential differences exist between the various surfaces. For a particular pair of conductors we label this potential difference ΔV. Since we can always set our zero of potential at our will, we can take one of the surfaces to have zero potential and the other to have a potential V which will equal ΔV. Therefore we will call the potential difference between the two surfaces V.

Let us consider the case of two isolated conductors (labeled 1 and 2) with charge $+Q$ on one and $-Q$ on the other, and a potential difference V between them. Depending on the shape of the conductors and their positions relative to each other, the charges on the conducting surfaces will distribute themselves with some definite (but not necessarily uniform) charge distribution, σ_1 and σ_2. In general, σ_1 and σ_2 will vary from point to point on the respective surfaces. In principle, the potential and electric field everywhere outside and on the conductors, can be determined by dividing the surfaces into tiny segments and calculating the potential (or electric field) at any point by adding the contributions of all the electric charges in all the tiny segments. It is not hard to see that if we doubled (or halved, or tripled) the electrical charges in *all* segments on both surfaces we would not disturb the equilibrium on those surfaces, and furthermore the potential and electric field everywhere would also double (or halve, or triple) as a consequence. This is equivalent to saying that if we doubled the total charges (Q and $-Q$) on both isolated conductors (and waited for equilibrium to return), the potential V between them would double (as would the surface charge distributions σ_1 and σ_2, everywhere on the surfaces). From this we conclude that V is proportional to Q, as long as the geometry stays the same. Thus, if for example we transfer charge between one conductor and the other, V would increase in proportion to the increases in $\pm Q$ on the surfaces. We can therefore write $V = (1/C)Q$, where $1/C$ is the constant of proportionality, or equivalently, $Q = CV$, and the constant C is called the **capacitance** of the system. This constant

C depends on the geometry of the conductors, their size, shape and position, but it does not depend on the charge on the plates. For any particular geometry we can calculate its capacitance by assuming a certain charge and calculating the resultant V. Then $C = Q/V$, and for any other Q this ratio remains the same. The unit for capacitance is the **farad** (F). A capacitance of one farad is very large, and more common capacitances are μF (10^{-6} F) or pF (10^{-9} F). If we build a unit containing two conductors with relatively large surfaces close to each other (but not touching) we call this object a **capacitor** whose capacitance is C. The name derives from the fact that C represents the capacity of the two conductors to store charge on their surfaces per unit potential difference (per volt) between them. A large capacitance means that the capacitor holds a lot of charge per volt, while a small capacitance means that only a small amount of charge is held per volt. We will first discuss the calculation of capacitances for several specific geometries, and the use of these results. Then we will discuss the energy needed to charge a capacitor and the interpretations of these results. The most common capacitor geometry is that of two close parallel, conducting plates.

Problem 4.23. A "parallel plate capacitor" consists of two parallel plates, of area A, separated by a small distance d and carrying charges of $\pm Q$ (as in Fig. 4-11). Assume that the field is uniform throughout the region between the plates.

(*a*) What is the field between the plates?

(*b*) What is the potential difference between the plates?

(*c*) What is the capacitance of this parallel plate capacitor?

Solution

(*a*) The field was calculated in Problem 3.23, and equals $E = \sigma/\varepsilon_0$. Ignoring edge effects, the surface charge, σ, is uniformly distributed and $\sigma = Q/A$, giving $E = Q/\varepsilon_0 A$. This is a uniform field pointing from the positive to the negative plate.

(*b*) As shown in Problem 4.8(*b*), the potential difference between the plates is just $V = Ed = \sigma d/\varepsilon_0 = Qd/\varepsilon_0 A$. The positive plate is at the higher potential.

(*c*) Using the results of (*b*), we get $C = Q/V = Q/(Qd/\varepsilon_0 A) = \varepsilon_0 A/d$.

Problem 4.23 shows that the capacity of a parallel plate capacitor can be written as

$$C = \varepsilon_0 A/d \text{ (parallel plate capacitor)} \tag{4.9}$$

Note. The capacitance (ability to hold, or store, charge per volt) increases in proportion to the cross-sectional area of the plates, A. Thus doubling the area doubles C. The capacitance also varies in inverse proportion to the separation distance, d. Thus halving d doubles C as well.

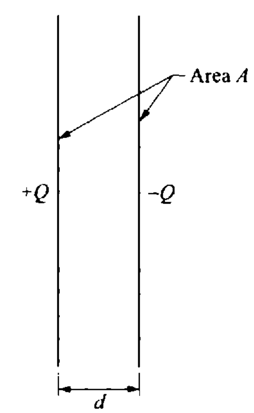

Fig. 4-11

Problem 4.24. A parallel plate capacitor has a capacitance of 2.5 μF and an area of 156 m^2.

(*a*) What is the distance between the plates?

(*b*) If one applies a voltage of 75 V to the capacitor how much charge is collected on each plate?

(*c*) How much work is needed to move an additional charge of 1.8×10^{-8} C from the negative to the positive plate?

Solution

(*a*) The capacitance is given by $C = \varepsilon_0 A/d = 2.5 \times 10^{-6} = (8.85 \times 10^{-12})(156\ \text{m}^2)/d$. Thus $d = 5.52 \times 10^{-4}$ m.

(*b*) The charge is given by $Q = CV = (2.5 \times 10^{-6}\ \text{F})(75\ \text{V}) = 1.88 \times 10^{-4}$ C.

(*c*) Since the charge we are moving is small compared to the charge already there the potential will hardly change as we move the charge. Therefore the work needed, which is just the increase in potential energy, will be given by $\Delta QV = (1.8 \times 10^{-8}\ \text{C})(75\ \text{V}) = 1.35 \times 10^{-6}$ J.

Problem 4.25. A parallel plate capacitor is built from plates with areas of 888 m^2 each and a separation of 1.6×10^{-4} m. The maximum electric field that can exist in the capacitor before the air ionizes causing sparking is 3.1×10^6 V/m.

(*a*) What is the capacitance of this capacitor?

(*b*) What is the maximum voltage that can be applied to this capacitor?

Solution

(*a*) The capacitance is given by $C = \varepsilon_0 A/d = (8.85 \times 10^{-12})(888\ \text{m}^2)/(1.6 \times 10^{-4}\ \text{m}) = 4.91 \times 10^{-5}$ F.

(*b*) The maximum electric field that the capacitor can stand before electrical breakdown is 3.1×10^6 V/m. The electric field is equal to $Q/\varepsilon_0 A = CV/\varepsilon_0 A = 3.1 \times 10^6$. Thus $V = (8.85 \times 10^{-12})\ (888\ \text{m}^2)\ (3.1 \times 10^6\ \text{V/m})/4.91 \times 10^{-5}\ \text{F} = 496$ V. This could have been derived more simply using the relationship that $V = dE$ for a uniform field, giving $V = (3.1 \times 10^6\ \text{V/m})(1.6 \times 10^{-4}\ \text{m}) = 496$ V.

Problem 4.26. A capacitor consists of two thin concentric hollow metal spherical shells of radii r_1 and r_2 ($r_1 < r_2$) with charges Q and $-Q$, respectively

(*a*) What is the capacitance of this capacitor?

(*b*) Show that all the charges reside on the outer surface of the inner shell and the inner surface of the outer shell.

Solution

(*a*) The potential produced by a uniform spherical shell of charge Q was calculated earlier and given by Eqs. (*4.5*): $V = (1/4\pi\varepsilon_0)Q/r$ for $r > R$ and $V = (1/4\pi\varepsilon_0)Q/R$ for $r < R$. On the outer surface of the outer spherical shell the potential is zero, since we are outside of each shell and the potential is therefore $V = V_1 + V_2 = kQ/r + k(-Q)/r = 0$, $r \geq r_2$. On the outer surface of the inner shell the potential from sphere two is still $-kQ/r_2$ but the potential from the first sphere is kQ/r_1. Thus $V = kQ(1/r_1 - 1/r_2)$, which is also the potential difference between the shells (since the potential at the second shell is zero). Then $C = Q/V = 4\pi\varepsilon_0/(1/r_1 - 1/r_2)$.

(*b*) Since the potential is constant everywhere in the outer shell and beyond (actually zero) the electric field is zero everywhere in this region. Since $E = \sigma/\varepsilon_0$ just outside a conducting surface, we have $\sigma = 0$ on the outside of the outer sphere, and all the charge, $-Q$, resides on the inside surface. Similarly, in the hollow region within the inner shell the potential is constant [(Eq. (*4.5*)] and the electric field again vanishes. Thus σ/ε_0 on the inner surface vanishes as well, and the entire charge Q resides on the outer surface of the inner shell.

Problem 4.27. The two shells of Problem 4.26 have radii of 1.6 m and 1.8 m.

(*a*) What is the capacitance of this arrangement?

(*b*) How much voltage must be applied across the shells to store a charge of 3.7×10^{-8} C on the shells?

Solution

(*a*) The capacitance was derived in the previous problem and equals $C = 4\pi\varepsilon_0/(1/r_1 - 1/r_2) = 4\pi(8.85 \times 10^{-12})/[1/1.6 \text{ m} - 1/1.8 \text{ m}) = 1.60 \times 10^{-9} \text{ F} = 1.6 \text{ nF}$.

(*b*) The charge is given by $Q = CV$, so $V = Q/C = (3.7 \times 10^{-8} \text{ C})/(1.6 \times 10^{-9} \text{ F}) = 23.1$ V.

4.7 COMBINATION OF CAPACITORS

Capacitors have many applications in electrical circuits, both using constant sources of voltage such as batteries (Chap. 3), and using time varying sources of voltage (Chap. 9) such as supplied by the electric utility. Often one uses combinations of capacitors and we inquire into the result of making such combinations. There are two basic different ways in which one can combine capacitors. The two are called series and parallel combinations. We will see later that the same types of combinations can be applied to resistors as well. In what follows we will assume that the pair of close conductors constituting a capacitor is sufficiently far from the conductors making up the next capacitor, that we do not have to worry about "cross-capacitance" between the two capacitors. In addition, all connections between capacitors are made with conducting wire, and the conductors and wires so connected must all be at the same potential when we have equilibrium. For visual simplicity we will carry out our discussion in the context of parallel plate capacitors.

First we discuss what is called the parallel connection of capacitors. Here one side of all the capacitors are kept at a common potential by being connected to each other by a conducting wire, while the other sides of all the capacitors are kept at a (different) common potential by connection to a second conducting wire. This is illustrated in Fig. 4-12. Here the two sides of C_1 (the symbol for a capacitor is —||—) are connected to points *a* and *b* by conducting wires and so are the two sides of capacitor C_2. If one has three capacitors in series one would connect C_3 between the same two points. The left sides of the capacitors are thus at a common potential, and the right sides are at a different common potential. The potential difference across each capacitor is the same, since in each case it will equal $V_a - V_b$. This

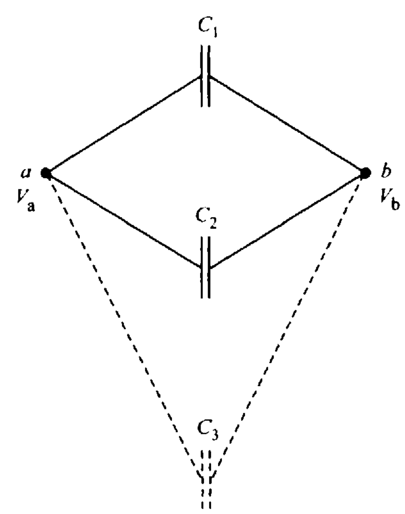

Fig. 4-12

is the defining characteristic of all parallel circuits: each branch has the same potential difference or voltage. We will use the next problem to develop the properties of a parallel circuit.

Problem 4.28. Consider the two capacitors in Fig. 4-12, connected between a difference of potential, $V = V_a - V_b$.

(*a*) What is the charge on the plates of each capacitor?

(*b*) What is the total charge collected on the equipotential surfaces connected to points a and b?

(*c*) If one replaced the two capacitors with a single capacitor, collecting the same charge between the two points, what capacitance would it have? (This is called the "equivalent" capacitor.)

(*d*) If $C_1 = 2.3\ \mu\text{F}$ and $C_2 = 5.7\ \mu\text{F}$, what is the equivalent capacitance of the combination?

Solution

(*a*) $Q_1 = C_1 V$ and $Q_2 = C_2 V$; i.e., if $V_a > V_b$, Q_1 and Q_2 will appear on the left plates of C_1 and C_2, respectively, while $-Q_1$ and $-Q_2$ will appear on the right plates of C_1 and C_2.

(*b*) The total charge is just the sum of Q_1 and Q_2 on side a and $-(Q_1 + Q_2)$ on side b.

(*c*) The equivalent capacitance would have to be charged to $(Q_1 + Q_2)$ when the potential difference across it is V. Thus, $C_{eq}V = Q_1 + Q_2 = C_1 V + C_2 V = (C_1 + C_2)V$. Dividing out by V we get:

$$C_{eq} = C_1 + C_2 \text{ (parallel combination)} \tag{4.10a}$$

(*d*) Using the given values for C_1 and C_2 we get $C_{eq} = (2.3 + 5.7)\ \mu\text{F} = 8.0\ \mu\text{F}$.

If capacitor C_3 were also connected as shown in Fig. 4-12 the same reasoning as in Problem 4.28 would lead to $C_{eq} = C_1 + C_2 + C_3$. In general, for any number of parallel capacitors,

$$C_{eq} = \sum C_i \tag{4.10b}$$

The other possible way to combine two capacitors is in series. Consider the two capacitors in Fig. 4-13. Here one plate of the first capacitor is connected to point a and the second plate is connected to the first plate of the next capacitor through point c. The second plate of the second capacitor is connected to point b. If there are more capacitors in series then the second is connected to the third and so on until the last is connected to point b. Now the potential across C_1 need not be the same as is the potential V_2 across C_2, since $V_1 = V_a - V_c$, and $V_2 = V_c - V_b$ and points a and b are not connected. Indeed the total voltage between a and b is $V = V_1 + V_2$. If we examine the figure more closely, we note if the first plate of C_1 accumulates charge $+Q_1$ (inserted or removed through point a), then the second plate of C_1 will have a charge of $(-Q_1)$. This follows because if it did not, the electric field immediately outside the plate would not vanish, and charges would flow in the wire (through point c) until the field vanished. This would occur when the charge is $-Q_1$. From where did this $-Q_1$ charge come? It must have come from the first plate of the second capacitor. In that case the second capacitor has the same charge as the first, $+Q_1$ on its first plate. Using the same reasoning as for the first capacitor, we conclude that the second capacitor will have charge $-Q_1$ on its second plate (where we presume that point b is connected to other parts of the circuit to or from which charges can flow). We are now ready

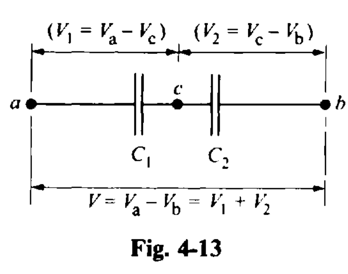

Fig. 4-13

to calculate the equivalent capacitance that we could use to replace C_1 and C_2.

Problem 4.29. Consider the two capacitors in series in Fig. 4-13. Calculate the equivalent capacitance.

Solution

We have just shown that each capacitor contains the same charge which we call Q. This is the charge which is supplied by the source of potential between a and b, and is the charge that will be on the equivalent capacitor that we can use to replace the combination of C_1 and C_2. Now $V_1 = Q/C_1$ and $V_2 = Q/C_2$. Then $V = V_1 + V_2 = Q(1/C_1 + 1/C_2) = Q/C_{eq}$. Thus

$$1/C_{eq} = 1/C_1 + 1/C_2 \text{ (series combination)} \tag{4.11a}$$

The same reasoning as used in Problem 4.29 can be used to generalize to any number of series capacitors:

$$1/C_{eq} = \sum (1/C_i) \tag{4.11b}$$

Often we have situations in which a number of capacitors are used in a circuit, some in series and some in parallel. In many cases we can combine the results of Eqs. (*4.10*) and (*4.11*) to obtain an overall equivalent capacitance.

Problem 4.30. Consider the combination of capacitors shown in Fig. 4-14(*a*). Here $C_1 = 2.5\ \mu$F, $C_2 = 3.5\ \mu$F, $C_3 = 5.6\ \mu$F and $C_4 = 1.3\ \mu$F.

(*a*) What is the equivalent capacitance of C_2 and C_3?

(*b*) What is the equivalent capacitance between points a and b?

(*c*) If a voltage of 10.5 V were provided between points a and b, what charge would accumulate on the equivalent capacitance?

(*d*) For case (*c*), what charge accumulates on capacitor C_1? On capacitor C_4?

(*e*) What charge accumulates on capacitor C_2? On capacitor C_3?

Solution

(*a*) Capacitors C_2 and C_3 are in parallel (points c and d play the role of points a and b of Fig. 4-12). They can therefore be replaced with an equivalent capacitance of $C_{eq} = C_2 + C_3 = (3.5 + 5.6)\ \mu\text{F} = 9.1\ \mu\text{F}$ [see Fig. 4-14(*b*)].

(*b*) If we replace C_2 and C_3 with an equivalent capacitance $C_{eq} = 9.1\ \mu$F, we then have three capacitors in series. Using Eq. (*4.11b*), we get the final equivalent capacitance to be $1/C_{f,eq} = 1/C_1 + 1/C_{eq} + 1/C_4 = 1/2.5 + 1/9.1 + 1/1.3 = 1.28$, and $C_{f,eq} = 0.78\ \mu$F [see Fig. 4-14(*c*)].

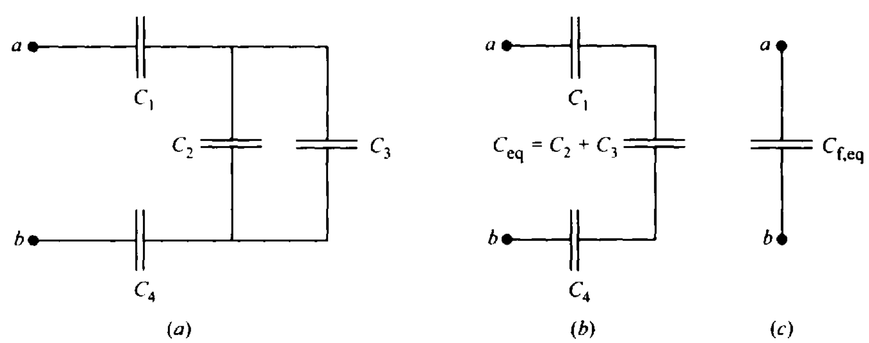

Fig. 4-14

(c) The voltage across $C_{f,eq}$ equals 10.5 V. Then the charge on the equivalent capacitor is $Q = C_{f,eq}V = 8.21 \times 10^{-6}$ C.

(d) In a series circuit, the charge on each capacitor is the same and is equal to the charge on the equivalent capacitor. Thus the charge on both C_1 and on C_4 is 8.21×10^{-6} C.

(e) The total charge on the two parallel capacitors C_2 and C_3 is the charge on C_{eq} which equals 8.21×10^{-6} C. This charge is distributed between C_2 and C_3. To get the individual charge Q_2 or Q_3 we need the voltage across each capacitor. We know that, for a parallel combination, the voltage across each capacitor is the same and is equal to the voltage across the equivalent capacitor. We can easily calculate the voltage across the equivalent capacitor $V' = Q/C_{eq} = (8.21 \times 10^{-6}\ \text{C})/(9.1\ \mu\text{F}) = 0.90$ V. Then $Q_2 = C_2V' = (3.5 \times 10^{-6}\ \text{F})(0.90) = 3.16 \times 10^{-6}$ C and $Q_3 = C_3V' = (5.6 \times 10^{-6}\ \text{F})(0.90) = 5.05 \times 10^{-6}$ C. Note that $Q_2 + Q_3 = 8.21 \times 10^{-6}$ C, as required.

4.8 ENERGY OF CAPACITORS

As stated previously, whenever we charge a capacitor we must do work to bring more positive charge to the plate that was already positively charged, and similarly to the negative plate. This work is converted into potential energy of the capacitor, which can be viewed as the energy stored by the charges that have been separated. As we will see, we can also take an alternative viewpoint that the effect of separating the charges is to produce an electric field in space, and that the accumulated energy is stored in these electric fields.

If a capacitor is charged to a difference of potential V, then the work by an outside force that is needed to transfer an additional small charge ΔQ from the negative to the positive plate is $(\Delta Q)V = Q(\Delta Q)/C$. Using arguments similar to those used to calculate the potential energy of the spring (Beginning Physics I, Problem 6.8), we can show that the work needed to accumulate a charge of Q on the capacitor is $W = (\frac{1}{2})Q^2/C$. Then the energy stored in a capacitor can be written as

$$U_p = (\tfrac{1}{2})Q^2/C = (\tfrac{1}{2})CV^2 = (\tfrac{1}{2})QV \qquad (4.12)$$

Problem 4.31. Derive the expression for the electrical potential energy stored in a capacitor C with charge Q [Eq. (4.12)]

Solution

We know that when the capacitor is charged to some value q_i the potential is given by $V_i = q_i/C$. The work necessary to bring the next increment of charge, Δq, across [so that the new plate charge will be $(q_i + \Delta q)$ and $-(q_i + \Delta q)$], is given by: $\Delta W_i = V_i \Delta q$. In Fig. 4-15 we show a plot of potential difference vs. charge for our capacitor, as well as the increment from q_i to $q_i + \Delta q$. Clearly, ΔW_i is just the area under the V vs. q curve between the adjacent dotted vertical lines. The total work done in bringing charges across, starting from $q = 0$ to $q = Q$ is just the triangular area under the V vs. q curve between the origin and $q = Q$. This is just: $W = (\frac{1}{2})QV = (\frac{1}{2})Q^2/C = (\frac{1}{2})CV^2$ as indicated in Eq. (4.10).

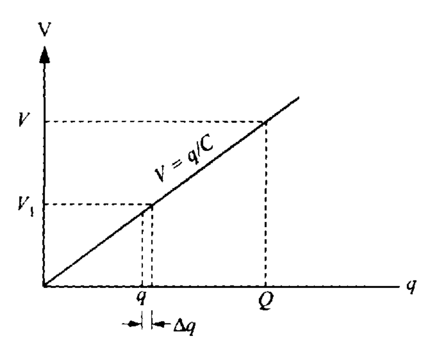

Fig. 4-15

Problem 4.32. A capacitor with $C = 82.3\ \mu F$ is charged to a voltage of 110 V.

(*a*) How much charge is accumulated on the capacitor?

(*b*) How much potential energy is stored in the capacitor?

(*c*) If the voltage on the capacitor is to be increased to 150 V, what additional work will have to be done?

(*d*) If the capacitor is discharged from 150 V to 75 V, how much work can be done by the electric field on the moving charges?

Solution

(*a*) The charge on the capacitor is $Q = CV = (82.3\ \mu F)(110\ V) = 9.05 \times 10^{-3}$ C.

(*b*) The potential energy is given by Eq. (*4.12*) as $U_p = (\frac{1}{2})CV^2 = (\frac{1}{2})(82.3 \times 10^{-6}\ F)(110\ V)^2 = 0.50$ J. (Alternatively, $U_p = (\frac{1}{2})Q^2/C = (\frac{1}{2})(9.05 \times 10^{-3}\ C)^2/(82.3\ \mu F) = 0.50$ J).

(*c*) The final potential energy is $(1/2)(82.3 \times 10^{-6})(150)^2 = 0.93$ J. The additional work is $W = \Delta U_p = U_{pf} - U_{pi} = 0.93 - 0.50 = 0.43$ J.

(*d*) When the electric field does positive work the electric potential energy decreases by a like amount. Thus $W = -\Delta U_p = U_{pi} - U_{pf} = 0.93\ J - (\frac{1}{2})(82.3 \times 10^{-6})(75)^2\ J = 0.70$ J.

Problem 4.33. Consider the combination of capacitors used in Problem 4.30, with the voltage of 10.5 V between points *a* and *b* (Fig. 4-14).

(*a*) What is the total potential energy stored in the combination?

(*b*) What is the energy stored on each of the capacitors?

Solution

(*a*) We showed that the equivalent capacitance of the combination between points *a* and *b* is 0.78 μF. Then the total energy stored is $(\frac{1}{2})C_{f,\,eq}V^2 = (\frac{1}{2})(0.78\ \mu F)(10.5)^2 = 4.3 \times 10^{-5}$ J.

(*b*) For each capacitor we can use either $U_p = (\frac{1}{2})CV^2$ or $U_p = (\frac{1}{2})Q^2/C$. On C_1 and C_4 we know that the charge is 8.21×10^{-6} C, so the energies are $U_{p1} = (\frac{1}{2})(8.21 \times 10^{-6})^2/2.5\ \mu F = 1.35 \times 10^{-5}$ J, and $U_{p4} = (\frac{1}{2})(8.21 \times 10^{-6})^2/1.3\ \mu F = 2.59 \times 10^{-5}$ J. For C_2 and C_3 we know that $V' = 0.90$ V. Thus $U_{p2} = (\frac{1}{2})3.5\ \mu F(0.90)^2 = 1.4 \times 10^{-6}$ J, and $U_{p3} = (\frac{1}{2})(5.6\ \mu F)(0.90)^2 = 2.3 \times 10^{-6}$ J. The total energy is then $(1.35 + 2.59 + 0.10 + 0.23) \times 10^{-5}\ J = 4.3 \times 10^{-5}$ J, as we found in part (*a*).

The energy that is stored in a capacitor can be viewed as the energy stored by the charge that has been separated. As a result of separating these charges, electric fields are established in space. We can therefore, alternatively, view the work done in separating the charges as the work required to produce these electric fields. The energy stored would then be viewed as the energy stored in these electric fields. We will illustrate this view by using a parallel plate capacitor as an example, but the result we derive will be valid for all situations in which electric fields are established.

Problem 4.34. Consider a parallel plate capacitor whose plates have an area of *A* and are separated by a distance *d*. As shown previously the capacitance is given as $C = \varepsilon_0 A/d$. A difference of potential *V* is established between the plates.

(*a*) Derive an expression for the energy stored in the capacitor in terms of the dimensions of the capacitor and the (constant) electric field within the capacitor.

(*b*) Derive an expression for the "**energy density**" (the energy per unit volume) within the capacitor.

Solution

(*a*) We know that the electric field within a parallel plate capacitor is $E = V/d$, and that the energy stored is $U_p = (\frac{1}{2})CV^2 = (\frac{1}{2})(\varepsilon_0 A/d)(Ed)^2 = (\frac{1}{2})(\varepsilon_0 E^2)(Ad)$.

(*b*) The volume within the capacitor is Ad. In this volume the electric field is given by the formula we used (again ignoring slight edge effects). Outside of this volume, the electric field is essentially zero. Thus the energy density is $U_{pd} = (\frac{1}{2})\varepsilon_0 E^2$. This is a general expression for the energy density (we will modify this slightly in the next section)

$$U_{pd} = (\tfrac{1}{2})\varepsilon_0 E^2 \tag{4.13}$$

Problem 4.35. A parallel plate capacitor has a capacitance of 2.6 μF. The plates are separated by a distance of 0.63 mm.

(*a*) If a voltage of 34 V is applied to the plates of the capacitor, calculate the energy stored in the capacitor.

(*b*) Calculate the electric field within the capacitor.

(*c*) Calculate the energy density within the capacitor.

(*d*) Use the results of parts (*a*) and (*b*) to obtain the area A of the capacitor plates

(*e*) Calculate the energy stored in a cylindrical volume of base area $A' = 0.36$ m^2 extending from one plate to the other within the capacitor.

Solution

(*a*) The energy stored is $(\frac{1}{2})CV^2 = (\frac{1}{2})(2.6\ \mu\text{F})(34\ \text{V})^2 = 1.50 \times 10^{-3}$ J.

(*b*) The electric field within the capacitor is $E = V/d = (34\ \text{V})/(0.63 \times 10^{-3}\ \text{m}) = 5.40 \times 10^4$ V/m.

(*c*) The energy density is given by $U_{pd} = (\frac{1}{2})(\varepsilon_0 E^2) = (\frac{1}{2})(8.85 \times 10^{-12})(5.40 \times 10^4)^2 = 1.29 \times 10^{-2}$ J/m^3.

(*d*) $U_p = U_{pd}(Ad) \rightarrow 1.5 \times 10^{-3}\ \text{J} = (1.29 \times 10^{-2}\ \text{J/m}^2)(0.63 \times 10^{-3}\ \text{m})A \rightarrow A = 185\ \text{m}^2$.

(*e*) The volume of the cylinder is $Ad = (0.36\ \text{m}^2)(0.63 \times 10^{-3}\ \text{m}) = 2.27 \times 10^{-4}$ m^3. The energy stored in that volume is the energy density times the volume $= 1.29 \times 10^{-2}(2.27 \times 10^{-4}) = 2.93 \times 10^{-6}$ J.

4.9 DIELECTRICS

So far we have discussed only cases in which charges establish electric fields and potentials in empty space or on conductors. If the region includes other, non-conducting materials, even when the materials are not charged (neutral), the atoms and molecules within that material may alter the fields that are otherwise produced. We have already seen that when neutral conductors are placed near free charges, the free charges in the conductors redistribute themselves on the surface and thereby produce fields of their own which must be added to the fields of the original charges. Unlike conductors other neutral materials do not have free charges and we must consider what mechanism might cause electrical effects to arise.

Normal **insulating materials** consist of atoms and molecules that are composed of positively charged nuclei and negatively charged electrons that are tightly bound together with no loose outer electrons that are free to roam. In the presence of an electric field the positive and negative charges in the atoms and molecules are pulled in opposite directions. As a result, the atoms and molecules will become somewhat "**polarized**" with the positive and negative charges becoming slightly separated from their equilibrium positions. This separation is expected to be approximately proportional to the magnitude of the electric field as long as the field is not too large. The (slightly) separated charges will produce their own electric field which must be added to the field established by the original charges. In general this can lead to many complications, and we will consider only a special case in which the effect can be easily understood.

Consider a parallel plate capacitor which is filled with some insulating material. We call this material a "**dielectric**" since, as we will show, it will produce its own electric field in a direction opposite to the original field. If we place a surface charge distribution of $\pm\sigma$ on the plates of the capacitor, this charge will produce an electric field of σ/ε_0 within the capacitor. The field will point from the positive to the negative plate. This field will cause a polarization of the material such that each atom will have its positive charge move closer to the negative plate (see Fig. 4-16). We will then have tiny "**dipoles**" throughout the material with positive charge to the left and negative charge to the right. In the interior of the dielectric the material remains uncharged since the shifting of negative charge slightly to the right from one parallel layer will be compensated by negative charge shifting into that layer from the next layer to the left. Only at the surfaces, next to the plates, will charge accumulate. On the left surface in Fig. 4-16 the electrons that shift to the right are not compensated for and a net positive charge appears; on the right surface negative charges moving from the layer just to the left of the surface accumulate on the surface, and cannot be compensated for by electrons moving further to the right. Since the bulk of the dielectric remains neutral, the net "polarization" charges on the two surfaces of the dielectric are equal and opposite. Thus the dielectric develops a surface charge next to each of the plates which is of opposite sign to the original charge on the plates. This is equivalent to an additional charge added to the plates which produces its own electric field in a direction opposite to the original field. The total field within the dielectric will therefore be reduced in this region. If the polarization is proportional to the field, then the new total field will be proportional to the field that would be produced in the absence of the dielectric material. We can then write that $E = E_0/\kappa$, where E is the total field in the presence of the dielectric, E_0 is the field that would be present without the dielectric and κ is the "**dielectric constant**" of the material. These dielectric constants vary from material to material, and some common examples are given in Table 4-1.

With this electric field the potential difference between the plates is $V = Ed = E_0 d/\kappa = \sigma d/\kappa\varepsilon_0 = Qd/A\kappa\varepsilon_0$, where σ and Q represent the free charge density and free total charge on the capacitor plates. Recalling that the capacitance without the dielectric is $C_0 = \varepsilon_0 A/d$, we have $V = Q/\kappa C_0 = Q/C$, where C is the true capacitance in the presence of the dielectric. Thus $C = \kappa C_0 = \kappa\varepsilon_0 A/d = \varepsilon A/d$, where $\varepsilon = \kappa\varepsilon_0$ is called the "**permittivity**" of the material. (correspondingly, ε_0 is called the **permittivity of free space.**) Since κ is always greater than 1, the addition of a dielectric within a capacitor increases the capacitance by the factor κ.

The energy stored in the capacitor is still given by $U_p = (\frac{1}{2})CV^2$, but both C and V are modified for a particular free charge Q on the plates. More charge is needed on each plate to produce the same

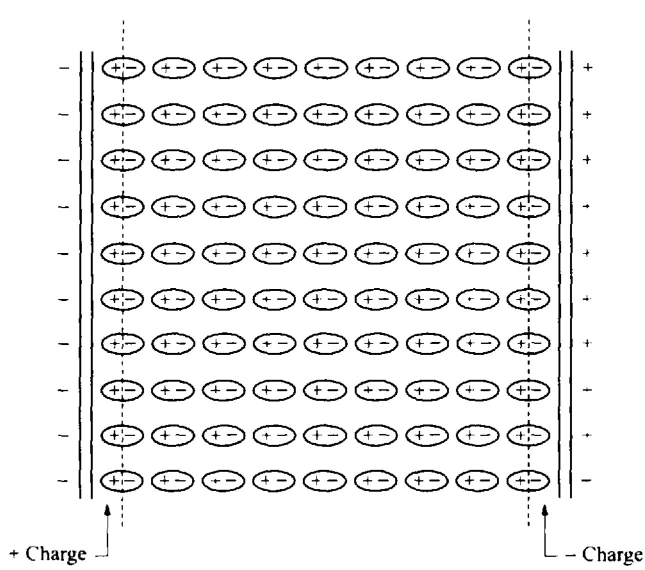

Fig. 4-16

Table 4.1. Dielectric Constants in Common Materials

Material	Dielectric constant
Vacuum	1
Air	1.0005
Teflon	2.1
Paper	3.3
Mica	3–6
Glass	5–10
Water	80.4

potential difference. Correspondingly, the energy density within the dielectric is modified from its value in vacuum, and is given by $U_{pd} = (\frac{1}{2})\varepsilon E^2$.

Problem 4.36. A parallel plate capacitor has plates with an area of 71 m^2. The plates are separated by a distance of 0.63 mm and the capacitor is filled with a dielectric of dielectric constant $\kappa = 2.6$. A voltage of 34 V is applied to the plates of the capacitor. Calculate (*a*) the capacitance of the capacitor, (*b*) the electric field within the capacitor, (*c*) the energy density within the capacitor, (*d*) the surface charge and charge density on the plates of the capacitor (the free charges) and (*e*) the surface charge and charge density on the dielectric layer near the plates.

Solution

(*a*) The capacitance is $C = \kappa\varepsilon_0 A/d = (2.6)(8.85 \times 10^{-12})(71\ \text{m}^2)/(0.63 \times 10^{-3}\ \text{m}) = 2.6\ \mu\text{F}$.

(*b*) The electric field within the capacitor is $E = V/d = (34\ \text{V})/(0.63 \times 10^{-3}\ \text{m}) = 5.40 \times 10^4\ \text{V/m}$.

(*c*) The energy density is given by $U_{pd} = (\frac{1}{2})(\varepsilon E^2) = (\frac{1}{2})(2.6 \times 8.85 \times 10^{-12})(5.40 \times 10^4)^2 = 3.35 \times 10^{-2}$ J/m^3.

(*d*) The charge on the plates is $Q = CV = 2.6\ \mu\text{F}(34\ \text{V}) = 8.84 \times 10^{-5}$ C. The charge density is $\sigma = Q/A$ $= 1.25\ \mu\text{C/m}^2$.

(*e*) The electric field within the capacitor is produced by two parallel charge distributions, that on the plates and that on the surface of the dielectric. Since the two distributions are of the opposite sign, the field produced is $E = (\sigma - \sigma_d)/\varepsilon_0 = (Q - Q_d)/A\varepsilon_0$. Now from part (*b*) $E = V/d = 5.4 \times 10^4$ V/m and $(\sigma - \sigma_0)/(8.85 \times 10^{-12}) = 5.4 \times 10^4 \rightarrow \sigma - \sigma_0 = 4.78 \times 10^{-7}\ \text{C/m}^2$. Recalling σ from part (*d*) we have $\sigma_d = 1.25\ \mu\text{C/m}^2 - 0.48\ \mu\text{C/m}^2 = 0.77\ \mu\text{C/m}^2$. The total surface charge on the dielectric is then $Q_d =$ $\sigma_d A = (0.77 \times 10^{-6})(71) = 5.47 \times 10^{-5}$ C.

Problem 4.37. A potential difference of 25 V is maintained across the plates of a parallel plate capacitor. The plates have an area of 43 m^2 and are separated by 1.56 mm.

(*a*) What is the capacitance of the capacitor if it is filled with air?

(*b*) How much energy is stored in this capacitor?

(*c*) What is the energy stored in the capacitor if it is filled with a dielectric of dielectric constant $\kappa = 1.9$ and the potential is held fixed?

(*d*) How much work is done when the dielectric is inserted between the plates?

(*e*) How much charge is on the plates with and without the dielectric?

Solution

(*a*) The capacitance is $C = \varepsilon_0 A/d = (8.85 \times 10^{-12})(43\ \text{m}^2)/(1.56 \times 10^{-3}\ \text{m}) = 0.244\ \mu\text{F}$.

(*b*) The energy stored $= (\frac{1}{2})CV^2 = (\frac{1}{2})(0.244\ \mu\text{F})(25\ \text{V})^2 = 7.62 \times 10^{-5}$ J.

(*c*) The energy stored is changed because the capacitance is increased to $\kappa C_0 = 1.9(0.244\ \mu\text{F}) = 0.464\ \mu\text{F}$. Then the energy stored is 1.44×10^{-4} J.

(*d*) The work done is the change in the energy stored, which equals $(1.44 - 0.76) \times 10^{-4}\ \text{J} = 6.8 \times 10^{-5}$ J. This work is done in the process of increasing the charge on the plates, as the dielectric is inserted, to keep the voltage across the capacitor fixed.

(*e*) The charge in each case equals $Q = CV$. For air, $Q = (0.244\ \mu\text{F})(25\ \text{V}) = 6.1 \times 10^{-6}$ C. For the dielectric, $Q = (0.464\ \mu\text{F})(25\ \text{V}) = 1.16 \times 10^{-5}$ C.

Problems for Review and Mind Stretching

Problem 4.38. A square, of side 0.38 m, has a charge $Q_1 = 7.6 \times 10^{-8}$ C at each of three corners, and a charge $Q_2 = -5.3 \times 10^{-8}$ C at the fourth corner, as in Fig. 4-17.

(*a*) What electric field is produced at the center of the square?

(*b*) What potential is produced at the center of the square?

(*c*) How much work must be done by an outside force to just remove Q_2 to a very large distance $(\rightarrow \infty)$?

Solution

(*a*) The magnitude of the field produced by each charge is $|E| = kQ/r^2$. The directions of E from the Q_1 at the two opposite corners are opposite and therefore cancel out. The direction of $\mathbf{E}_1$ is toward q_2, and has a magnitude of $|E_1| = (9.0 \times 10^9)(7.6 \times 10^{-8}\ \text{C})/(0.38/\sqrt{2}\ \text{m})^2 = 9.47 \times 10^3$ V/m. The direction of $\mathbf{E}_2$ is also toward q_2 since Q_2 is negative, and has a magnitude of $|E_2| = (9.0 \times 10^9)(5.3 \times 10^{-8}\ \text{C})/(0.38/\sqrt{2}\ \text{m})^2 = 6.61 \times 10^3$ V/m. The sum of these two fields is toward Q_2, and equals $(9.47 + 6.61) \times 10^3\ \text{V/m} = 1.61 \times 10^4$ V/m. This is the total field at the center of the square.

(*b*) The potential at the center is the scalar sum of the potential due to each charge. It therefore equals $V = 3V_1 + V_2 = k(3Q_1 + Q_2)/r = (9.0 \times 10^9)(3 \times 7.6 - 5.3) \times 10^{-8}\ \text{C}/(0.38/\sqrt{2}\ \text{m}) = 5.86 \times 10^3$ V.

(*c*) To calculate the work needed to remove Q_2 far away, we must calculate the change in potential energy between the case of Q_2 at infinity and at its present position. The change that occurs is that the potential energy between Q_2 and the three other charges becomes zero at ∞, while the potential energy between the fixed three charges does not change. When Q_2 is at its present position its potential energy equals the sum of kQ_1Q_2/r_{12} for each of the three charges. Two of the charges are at a distance of 0.38 m from Q_2, and the third charge is at a distance of $0.38\sqrt{2}$ m from Q_2. Thus $U_{\text{pi}} = (9.0 \times 10^9)(7.6 \times 10^{-8}\ \text{C})(-5.3 \times 10^{-8}\ \text{C})(2/0.38\ \text{m} + 1/0.38\sqrt{2}\ \text{m}) = -2.58 \times 10^{-4}$ J. The change in potential energy, which is the work that is needed, is 2.58×10^{-4} J.

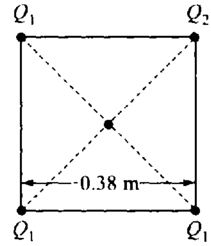

Fig. 4-17

Problem 4.39. A dipole consists of a positive charge q at $x = d/2$ and a negative charge $-q$ at $x = -d/2$ (as in Fig. 4-18). The dipole "moment", p, is defined as $p = qd$, where d is the distance between the charges.

(a) What is the potential produced by this dipole at a point on the x axis far from the dipole, i.e. at $x \gg d$?

(b) What is the potential produced by this dipole at a point on the y axis?

(c) What is the potential produced by this dipole at a point (x,y) far from the dipole, i.e. $r = (x^2 + y^2)^{1/2} \gg d$?

Solution

(a) The potential is the sum of the potential from the two charges. Thus $V = kq/(x - d/2) - kq/(x + d/2)$. Combining by using the common denominator gives, $V = kq[(x + d/2) - (x - d/2)]/[(x + d/2)(x - d/2)] = kqd/(x^2 - d^2/4) \approx k(qd)/x^2 = kp/x^2$, since $x \gg d$. In the numerator we were unable to neglect d compared to x, because the x canceled upon subtraction and we are left with d as a multiplicative factor, but in the denominator the x^2 term clearly dominates.

(b) In this case the potential is $V = kq/[y^2 + (d/2)^2]^{1/2} - kq/[y^2 + (-d/2)^2]^{1/2} = 0$.

(c) The distance from the charges to the point (x,y) is $[(x - d/2)^2 + y^2]^{1/2}$ and $[(x + d/2)^2 + y^2]^{1/2}$ for the positive and negative charges, respectively. For $r \gg d$, each of these is approximately equal to $r = (x^2 + y^2)^{1/2}$, and we can use this approximation whenever we are not subtracting the two distances from each other. We can write $V = kq[1/[(x - d/2)^2 + y^2]^{1/2} - 1/[(x + d/2)^2 + y^2]^{1/2}]$. Combining using a common denominator we get $V = [kq/r^2]\{[(x + d/2)^2 + y^2]^{1/2} - [(x - d/2)^2 + y^2]^{1/2}\}$, where we have used the approximation that $[(x \pm d/2)^2 + y^2]^{1/2} \approx (x^2 + y^2)^{1/2} = r$ in the denominator. Now, $[(x + d/2)^2 + y^2]^{1/2} = [x^2 + dx + d^2/4 + y^2]^{1/2} \approx [r^2 + dx]^{1/2} \approx r(1 + dx/2r^2)$. Similarly, $[(x - d/2)^2 + y^2]^{1/2} = [x^2 - dx + d^2/4 + y^2]^{1/2} \approx [r^2 - dx]^{1/2} \approx r(1 - dx/2r^2)$. Then $V \approx (kq/r^2)[(r + dx/2r) - (r - dx/2r)] = kqdx/r^3 = kp \cos\theta/r^2$. This result gives us the correct answer for part (a) when $\theta = 0$ and for part (b) when $\theta = 90°$.

Problem 4.40. A charge of $Q_1 = 4.35 \times 10^{-8}$ C is at the center of a conducting spherical shell of inner radius $r_1 = 0.93$ m and outer radius $r_2 = 1.07$ m. The shell itself has a charge of $Q' = -7.55 \times 10^{-8}$ C.

(a) What charge Q_1 is on the inner surface of the sphere and what charge Q_2 is on the outer surface?

(b) What is the potential at $r = 1.55$ m?

(c) What is the potential at $r = 1.00$ m?

(d) What is the potential at $r = 0.67$ m?

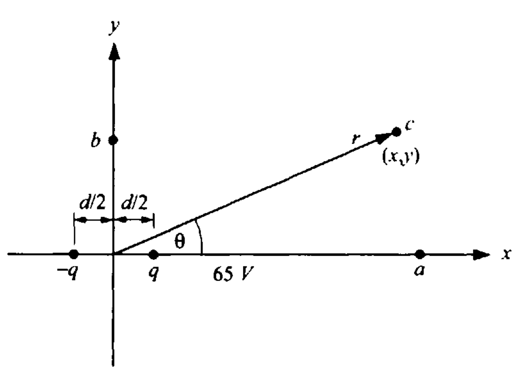

Fig. 4-18

Solution

(*a*) The charge on the inner surface must equal $Q_1 = -Q = -4.35 \times 10^{-8}$ C in order that the field is zero within the conducting sphere. Then the charge on the outer surface must equal $Q_2 = -3.20 \times 10^{-8}$ C so that the total charge on the shell is $Q' = -7.55 \times 10^{-8}$ C.

(*b*) The potential at any point is the sum of the potential produced by the three charges: Q, Q_1 and Q_2. The potential from Q is kQ/r, and the potential from the charges on the surfaces is given by Eq. (*4.5*): (a) $V = (1/4\pi\varepsilon_0)Q/r$ for $r > R$ and (*b*) $V = (1/4\pi\varepsilon_0)Q/R$ for $r < R$. At $r = 1.55$ m we are outside of all the charge distributions, and the total potential is $V = k(Q + Q_1 + Q_2)/r = (9.0 \times 10^9)(4.35 - 4.35 - 3.20) \times 10^{-8}\ \text{C}/1.55 = -186$ V.

(*c*) At $r = 1.00$ m, we are outside of Q and Q_1, but inside Q_2. Then $V = k(Q + Q_1)/r + kQ_2/r_2 = 0 + (9.0 \times 10^9)(-3.20 \times 10^{-8}\ \text{C})/1.07 = -269$ V.

(*d*) At $r = 0.67$ m, we are inside Q_1 and Q_2, and $V = kQ/r + k(Q_1/r_1 + Q_2/r_2) = (9.0 \times 10^9)(4.35 \times 10^{-8}\ \text{C})/0.67\ \text{m} + (9.0 \times 10^9)(-4.35 \times 10^{-8}\ \text{C}/0.93\ \text{m} - 3.20 \times 10^{-8}\ \text{C}/1.07\ \text{m}) = -106$ V.

Problem 4.41. The capacitance of two concentric spherical shells was calculated in Problem 4.26 as $C = 4\pi\varepsilon_0/(1/r_1 - 1/r_2)$. Show that as $r_1 \to r_2$, the capacitance approaches $\varepsilon_0 A/d$, where A is the surface area of the sphere and $d = r_2 - r_1$.

Solution

The capacitance can be written as $C = 4\pi\varepsilon_0 r_1 r_2/(r_2 - r_1)$. As $r_1 \to r_2$, $C \to 4\pi\varepsilon_0 r^2/d = \varepsilon_0 A/d$. This is just the formula for a parallel plate capacitor of area A separated by d. Thus the two spherical surfaces behave like two parallel surfaces separated by d.

Problem 4.42. A coaxial cable consists of an inner conducting cylinder of radius r_1 and a coaxial conducting cylindrical shell of inner radius r_2. Calculate the capacitance between the inner and outer cylinders for one meter of this cable.

Solution

We assume that the inner cylinder has a charge of $+Q$ and the outer cylinder has a charge of $-Q$. To calculate the potential difference between the cylinders we make use of the formulas given for charged cylinders in Eq. (*4.7*) for a long cylinder with surface charge at R: (*a*) $V = -(\lambda/2\pi\varepsilon_0)\ln(r/R')$ for $r > R$; (*b*) $V = -(\lambda/2\pi\varepsilon_0)\ln(R/R')$ for $r < R$. Here $\lambda = Q/L$, and R' is an arbitrary radius, usually taken as R. The potential at r_2 will then equal $V = V_1 + V_2 = 0$, since we get opposite contributions from the two surface charges using Eq. (*4.7a*) for both. At r_1, we must use Eq. (*4.7b*) for V_2, since we are now at $r < r_2$. Then $V = -(\lambda/2\pi\varepsilon_0)\ln(r_1/R') - (-\lambda/2\pi\varepsilon_0)\ln(r_2/R') = (\lambda/2\pi\varepsilon_0)\ln(r_2/r_1) = (Q/2\pi\varepsilon_0 L)\ln(r_2/r_1)$. The capacitance per unit length $C/L = Q/VL = 2\pi\varepsilon_0/\ln(r_2/r_1)$.

Problem 22.43. Several capacitors are connected as in Fig. 4-19(*a*). The capacitors have capacitance of: $C_1 = C_6 = 2.5\ \mu\text{F}$, $C_2 = C_3 = C_4 = 1.5\ \mu\text{F}$, $C_5 = 3.5\ \mu\text{F}$. The charge on C_3 is $Q_3 = 5.3 \times 10^{-6}$ C.

(*a*) What is the equivalent capacitance between points a and f?

(*b*) What is the difference of potential between points c and d?

(*c*) What is the difference of potential between points b and e?

(*d*) What is the difference of potential between points a and f?

(*e*) What is the charge on each capacitor?

Solution

(*a*) We first calculate the equivalent capacitance of the three capacitors that are in series, C_2, C_3 and C_4. This is given by $1/C_{eq} = 1/C_2 + 1/C_3 + 1/C_4 = 3(1/1.5\ \mu\text{F})$, or $C_{eq} = 0.50\ \mu\text{F}$. The circuit can then be

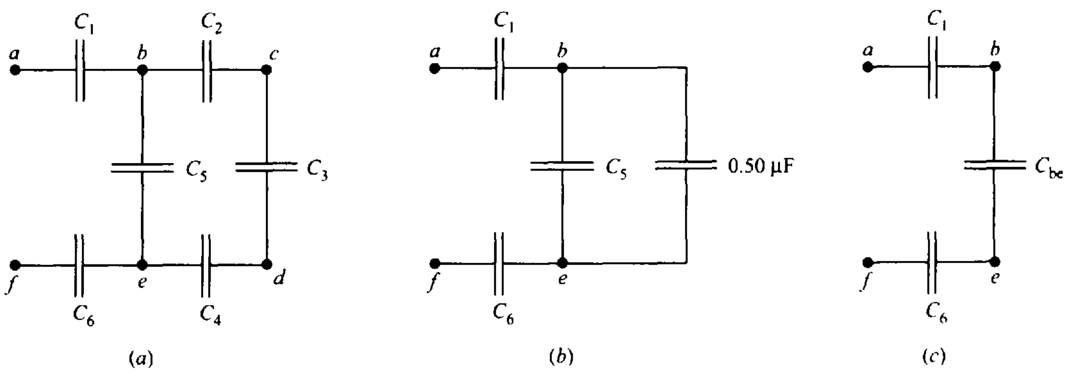

Fig. 4-19

redrawn as in Fig. 4-19(*b*). We then combine this capacitor and the parallel capacitor C_5 with an equivalent capacitor C_{be} of $C_{be} = (3.5 + 0.5)\ \mu\text{F}$, as in Fig. 4-19(*c*). Finally, we combine the three series capacitors in this figure to get the equivalent capacitance between *a* and *f*, $C_{af} = 1/C_1 + 1/C_{be} + 1/C_6$, giving $C_{af} = 0.952\ \mu\text{F}$.

(*b*) The potential difference between the points *c* and *d* is the potential across $C_3 = Q_3/C_3 = (5.3 \times 10^{-6}\ \text{C})/(1.5 \times 10^{-6}\ \text{F}) = 3.53\ \text{V}$.

(*c*) The potential difference between the points *b* and *e* is the potential across each of the parallel capacitors Fig. 4-19(*b*). The charge on the 0.50 μF capacitor is the same as on each of the three series capacitors, C_2, C_3 and C_4, which is 5.3×10^{-6} C. Thus $V_{be} = (5.3 \times 10^{-6}\ \text{C})/(0.50 \times 10^{-6}\ \text{F}) = 10.6\ \text{V}$.

(*d*) The potential difference V_{af} will equal Q/C_{af} where Q is the common charge on each of the three series capacitors in Fig. 4-19(*c*). The charge on C_{be} can be calculated as $C_{be}V_{be} = (4.0 \times 10^{-6}\ \text{F})(10.6\ \text{V}) = 4.24 \times 10^{-5}\ \text{C}$. Then $V_{af} = (4.24 \times 10^{-5}\ \text{C})/(0.952 \times 10^{-6}\ \text{F}) = 44.5\ \text{V}$.

(*e*) In part (*d*) we already used the fact that $Q_1 = Q_6 = Q_{be} = 4.24 \times 10^{-5}$ C [Fig. 4-19(*c*)]. From Fig. 4-19(*a*) we see that $Q_2 = Q_3 = Q_4 = 5.3 \times 10^{-6}$ C. From Fig. 4-19(*b*) we see that $Q_5 = C_5 V_{be} = (3.5 \times 10^{-6}\ \text{F})(10.6\ \text{V}) = 3.71 \times 10^{-5}\ \text{C}$.

Problem 4.44. In a certain region of space the equipotential surfaces are the surfaces of concentric spheres. The potential is given as $V = -V_0 r/r_0$, where $V_0 = 38$ V, is the potential at $r_0 = 0.35$ m and r is the distance from the center of the concentric spheres.

(*a*) What is the direction of the electric field at a distance r from the center of the spheres?

(*b*) What is the magnitude of the field at this value of r?

(*c*) If a particle with a charge of 6.1×10^{-7} C and mass 9.3×10^{-15} kg has a speed of 3.8×10^5 m/s at $r = 0.35$ m, what is the speed of this particle when it reaches $r = 2.8$ m?

Solution

(*a*) The electric field lines are always perpendicular to the equipotential surface and point from high to low potential. The direction that is perpendicular to the surface of concentric spheres is the radial direction. Therefore the field points along a radius. Since the potential decreases as r increases (it becomes more negative), the field points away from the center (outward) along the radius.

(*b*) The magnitude of the electric field is given by $|E| = |\Delta V/\Delta d|$ when Δd is along the direction of the field lines. To get $|E|$ we calculate V at r and at $(r + \Delta r)$ and subtract to get ΔV. This gives us $|\Delta V| = (V_0/r_0)[(r + \Delta r) - r] = V_0 \Delta r/r_0$. Thus $|E| = \Delta V/\Delta r = V_0/r_0$, and the magnitude of E is constant throughout the region.

(c) We use conservation of energy in this part. This requires that the sum of the potential and kinetic energy be the same at both points. The potential energy is $U = qV$ and the kinetic energy is $K = (\frac{1}{2})mv^2$. Initially $K = (\frac{1}{2})(9.3 \times 10^{-15}\ \text{kg})(3.8 \times 10^5\ \text{m/s})^2 = 6.71 \times 10^{-4}$ J, and $U_p = q(-V_0) = (6.1 \times 10^{-7}\ \text{C})(-38\ \text{V}) = -2.32 \times 10^{-5}$ J. At $r = 2.8$ m, $U_p = q(-V_0 r/r_0) = (6.1 \times 10^{-7}\ \text{C})(-38 \times 2.8/0.35) = -1.85 \times 10^{-4}$ J. Then adding kinetic and potential energies, $6.71 \times 10^{-4} - 2.32 \times 10^{-5} = -1.85 \times 10^{-4}\ \text{J} + K$ giving $K = 8.33 \times 10^{-4}$ J. Then $v_f = 4.23 \times 10^5$ m/s.

Note. Newton's 2nd law could be easily used to get this result only if the initial velocity were along a radius. Our result is quite general.

Problem 4.45. A charge Q produces an electric field of magnitude $|E| = kQ/r^2$. How much energy is stored by this electric field in a spherical shell at radius r and thickness Δr, where $\Delta r \ll r$?

Solution

Within this shell the electric field can be considered constant since r hardly varies. The energy density is given by $U_{pd} = (\frac{1}{2})\varepsilon_0 E^2 = (\frac{1}{2})\varepsilon_0[(1/4\pi\varepsilon_0)Q/r^2]^2$. For a thin shell the volume is equal to the surface area of the shell times the thickness of the shell, or volume $= 4\pi r^2 \Delta r$. The energy stored equals $U_{pd} \times$ volume $= Q^2 \Delta r / 8\pi\varepsilon_0 r^2$.

Problem 4.46. A parallel plate capacitor C is given a charge Q with air between the plates. The capacitor is then isolated so that no charge can be added or removed from the plates. Then a dielectric, of dielectric constant κ, is inserted between the plates, filling $\frac{1}{3}$ of the volume (see Fig. 4-20).

(a) What is the potential difference between the plates when there is air between the plates?

(b) What is the potential difference between the plates when the dielectric material is between the plates?

(c) What is the capacitance of the plates when the dielectric material is between the plates?

Solution

(a) The potential difference is $V = Q/C$.

(b) The electric field is now produced by the charges on the plates and also by the polarization surface charges on the dielectric material. The charge on the dielectric material does not produce any field in the region outside of the dielectric since the two surfaces are oppositely charged and they add to zero outside the material. Within the material (as discussed in the text for the case of dielectric filling the

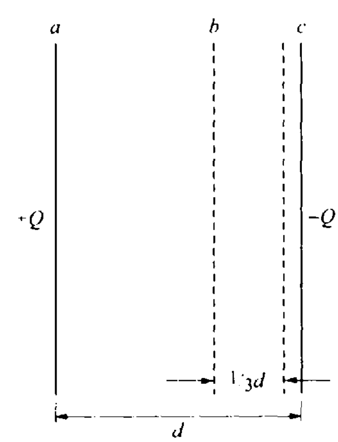

Fig. 4-20

entire space) the electric field will be reduced to E_0/κ, where $E_0 = Q/\varepsilon_0 A$, the field for the dielectric free capacitor. If we now move along a line from the positive plate to the negative plate, the potential difference from a to b is $E_0(2d/3)$, and the potential difference from b to c is $(E_0/\kappa)(d/3)$. Then $V = E_0 d(2/3 + 1/3\kappa) = (Qd/\varepsilon_0 A)(2/3 + 1/3\kappa) = Q(2/3 + 1/3\kappa)/C$.

(*c*) The new capacitance is $C' = Q/V = C/(2/3 + 1/3\kappa)$.

Supplementary Problems

Problem 4.47. A charge of 6.8×10^{-7} C is at a distance of 0.96 m from a second charge. The potential energy of the combination is -3.8×10^{-3} J. What is the charge on the other charge?

Ans. -6.0×10^{-7} C

Problem 4.48. Three charges are at the corners of an equilateral triangle of side 2.5 cm. The charges have charge of 5.3×10^{-8} C, -6.9×10^{-8} C and -9.9×10^{8} C. What is the total potential energy of the combination?

Ans. -7.5×10^{-4} J

Problem 4.49. Two charges of $q = 5.6 \times 10^{-7}$ C are located on the x axis at $x = \pm 0.76$ m.

(*a*) What is the potential at $x = 1.52$ m on the x axis?

(*b*) What is the potential at $x = -1.52$ m an the x axis?

(*c*) What is the potential at $y = 1.52$ m on the y axis?

(*d*) What is the potential at the origin, $x = y = 0$?

Ans. (*a*) 8.84×10^3 V; (*b*) 8.84×10^3 V; (*c*) 5.93×10^3 V; (*d*) 1.33×10^4 V

Problem 4.50. A charge of 5.3×10^{-7} C is located at the origin and a second charge of -4.5×10^{-7} C is on the x axis at $x = 2.1$ m. At what two points on the x axis is the potential equal to 500 V? (Refer to Problem 4.5 for a similar problem.)

Ans. $x = 1.073$ m and $x = -4.15$ m

Problem 4.51. A charge of 4.5×10^{-7} C is at $x = -0.19$ m and a charge of -5.3×10^{-7} C is at $x = +0.19$ m. At what point or points on the y axis is the potential equal to -500 V?

Ans. $y = \pm 1.43$ m

Problem 4.52. A ring of uniformly distributed charge has a radius of 1.81 m and contains a total charge of 6.5×10^{-7} C.

(*a*) At what distance from the plane of the ring is the potential equal to 1100 V along the axis of the ring?

(*b*) How much work must be done to move a charge of 3.8×10^{-7} C from this point to the center of the ring?

Ans. (*a*) 5.0 m; (*b*) 8.10×10^{-4} J

Problem 4.53. A large plane sheet has a surface charge density of 3.7×10^{-8} C/m^2. Point a is at a distance of 2.1 cm to the left of the sheet, point b is 1.1 cm to the left, point c is 1.1 cm to the right and point d is 2.1 cm to the right of the sheet.

(a) What is the potential difference between points a and b, $V_a - V_b$?

(b) What is the potential difference between points b and c, $V_b - V_c$?

(c) What is the potential difference between points c and d, $V_c - V_d$?

Ans. (a) -20.9 V; (b) 0; (c) 20.9 V

Problem 4.54. Two large parallel plane sheets are uniformly charged and separated by 5.6 cm. The sheet on the left has a surface charge density of 3.7×10^{-8} C/m^2 and the one on the right has a surface charge density of -1.3×10^{-8} C/m^2. Point a is between the sheets at a distance of 1.2 cm from the left sheet, point b is between the sheets at a distance of 1.2 cm from the right sheet and point c is to the right of both sheets at a distance of 1.2 cm from the right sheet.

(a) What is the potential difference between points a and b, $V_b - V_a$?

(b) What is the potential difference between points c and b, $V_c - V_b$?

Ans. (a) 90.6 V; (b) 50.2 V

Problem 4.55. A charge of 6.2×10^{-7} C is at the center of a charged conducting spherical shell of inner radius 0.86 m and outer radius 0.91 m. At a distance of 1.00 m from the charge, the potential is 4.92×10^3 V.

(a) What charge is on the sphere?

(b) What is the potential on the surface of the sphere?

(c) What is the potential at a point within the sphere at a distance of 0.50 m from the central charge?

Ans. (a) -7.33×10^{-8} C; (b) 5.41×10^3 V; (c) 1.01×10^4 V

Problem 4.56. A charge Q_1 is at the center of a charged conducting spherical shell of inner radius 0.54 m and outer radius 0.77 m that has a charge Q_2. At a point 0.40 m from the central charge, the potential is 985 V and on the sphere the potential is 880 V.

(a) What is the charge Q_1?

(b) What is the charge Q_2?

Ans. (a) 1.80×10^{-8} C; (b) 5.72×10^{-8} C

Problem 4.57. A long straight wire has a uniform charge of 6.3×10^{-9} C/m. What is the difference of potential between a point a which is 0.62 m to the left of the wire and a point b that is 0.13 m to the right of the wire, i.e. what is $V_a - V_b$?

Ans. -177 V

Problem 4.58. Two long wires are parallel to each other, separated by a distance of 0.43 m, and have uniform charges of 1.9×10^{-9} C/m and -7.3×10^{-9} C/m, respectively. Point a is midway between the wires and point b is 0.20 m from the negatively charged wire (and 0.63 m from the positively charged wire). What is the difference of potential $V_a - V_b$?

Ans. 46.3 V

Problem 4.59. Two long wires are each uniformly charged, with one along the x axis and the other along the y axis. The one along the x axis has a charge of 1.9×10^{-9} C/m, and the one along the y axis has a charge of 2.5×10^{-9} C/m. Point a is at (0.15, 0.15), point b is at (0.45, 0.15), point c is at (0.15, 0.45) and point d is at (0.45, 0.45).

(*a*) What is the potential difference $V_b - V_a$?

(*b*) What is the potential difference $V_c - V_a$?

(*c*) What is the potential difference $V_d - V_a$?

Ans. (*a*) 49.4 V; (*b*) 37.6 V; (*c*) 87.0 V

Problem 4.60. A long straight line carries a uniform charge of 6.6×10^{-9} C/m. A long conducting cylindrical shell, carrying a charge of -4.8×10^{-9} C/m is coaxial with the line and has an inner radius of 0.25 m and an outer radius of 0.27 m. Use $R = 0.25$ m for calculating the potential.

(*a*) What is the linear charge density on the inner and on the outer surface of the cylinder?

(*b*) What is the potential at $r = 0.36$ m?

(*c*) What is the potential at $r = 0.27$ m, the surface of the cylinder?

(*d*) What is the potential at $r = 0.15$ m?

Ans. (*a*) -6.6×10^{-9} C/m and 1.8×10^{-9} C/m; (*b*) -11.8 V; (*c*) -2.5 V; (*d*) 58.2 V

Problem 4.61. A long wire has a uniform positive charge distribution along its length.

(*a*) What are the equipotential surfaces for this wire?

(*b*) In which direction does the electric field point?

Ans. (*a*) cylindrical surfaces coaxial with the wire; (*b*) radially outward

Problem 4.62. A long straight wire carries a charge of 4.9×10^{-7} C/m. A short segment of insulating wire, of length 0.077 m, is parallel to the long wire, and carries a total charge of 6.8×10^{-6} C. How much work is needed to move this short wire from a distance of 5.3 m to 3.1 m from the long wire?

Ans. 3.22×10^{-2} J

Problem 4.63. A dipole is at the origin, oriented along the x axis. The dipole moment is 6.7×10^{-9} C · m, with the positive charge on the positive x side. Two charges of $\pm 5.0 \times 10^{-6}$ C are separated by a distance of 0.39 m and placed along the x axis with the positive charge nearer the dipole at a distance of 2.10 m. Refer to Problem 4.39 for the potentials.

(*a*) What is the potential energy of the charges in this position?

(*b*) If the charges are rotated by 90° and shifted so that the charges are now both at $x = 2.10$ m and $y = \pm 0.195$ m, what is the potential at this position?

(*c*) How much work by an outside force was done to turn the charges?

Ans. (*a*) 1.97×10^{-5} J; (*b*) 0; (*c*) -1.97×10^{-6} J

Problem 4.64. A certain charge distribution gives a potential of $V = -A/r^4$, where A is a positive constant and r is the distance from the origin.

(*a*) What are the equipotential surfaces for this potential?

(*b*) In which direction does the electric field point?

(*c*) What is the magnitude of the electric field? (*Hint*: See Problem 4.13)

Ans. (*a*) spherical surfaces centered on the origin; (*b*) radially in; (*c*) $4/r^5$

Problem 4.65. A proton has a speed of 6.0×10^6 m/s. The mass of a proton is 1.67×10^{-24} kg, and the charge is the same as on an electron (except that it is positive).

(*a*) What is the kinetic energy of the proton in Joules and in eV?

(*b*) If all the kinetic energy was gained by falling through a difference of potential, what difference in potential is required?

Ans. (*a*) 3.01×10^{-14} J = 1.88×10^5 eV; (b) 188 keV

Problem 4.66. An electron is moving with constant speed in a circle around a proton. The centripetal force is supplied by the electrical force between the proton and the electron. The radius of the orbit is $r = 0.53 \times 10^{-10}$

(*a*) What is the potential energy of the system in eV?

(*b*) Use the equation relating the (mass) × (centripetal acceleration) to the electrical force to deduce the kinetic energy of the electron in eV directly from the result of (*a*).

(*c*) What is the total energy of the system in eV?

(*d*) How much energy is needed to ionize the system, i.e. to remove the electron to a position at rest at infinity (total energy equal to zero)?

Ans. (*a*) −27.2 eV; (*b*) 13.6 eV; (*c*) −13.6 eV; (*d*) 13.6 eV

Problem 4.67. A particle, of mass 1.8×10^{-27} kg and charge 1.6×10^{-19} C is fixed to the origin. Another charge, of mass 9.1×10^{-31} kg and charge -1.6×10^{-19} C is initially at a distance of 9.3×10^{-10} m from the origin and moving directly away from the origin with a speed of 5.14×10^5 m/s. At what distance from the origin does this second particle stop and reverse its direction?

Ans. 1.8×10^{-9} m

Problem 4.68. A capacitor is built out of two closely spaced concentric spherical shells separated by a distance of 0.83 mm. The capacitance is 25 nF. What is the radius of the shells? (Refer to Problem 4.41.)

Ans. 0.43 m

Problem 4.69. A certain capacitor has an electric field of 2.85×10^5 V/m when 120 V are across the capacitor.

(*a*) What is the distance between the plates?

(*b*) If the area of the plates is 33 m^2, what is the capacitance of the capacitor?

(*c*) What is the energy in the capacitor when the voltage across the capacitor is 120 V?

(*d*) What is the electrical energy density in the capacitor at this voltage?

Ans. (*a*) 0.42 mm; (*b*) 0.69 μF; (*c*) 5.0×10^{-3} J; (*d*) 0.359 J/m^3

Problem 4.70. Four capacitors are connected in series and a voltage of 12 V is connected across the circuit. The capacitances are 1.3 μF, 2.5 μF, 6.8 μF and 0.92 μF.

(*a*) What is the equivalent capacitance of the circuit?

(*b*) What is the voltage across each capacitor?

(*c*) What is the total energy stored in the system?

Ans. (*a*) 0.416 μF; (*b*) 3.84 V, 2.00 V, 0.73 V, 5.42 V; (*c*) 3.0×10^{-5} J

Problem 4.71. Four capacitors are connected in parallel and a voltage of 12 V is connected across the circuit. The capacitances are 1.3 μF, 2.5 μF, 6.8 μF and 0.92 μF.

(*a*) What is the equivalent capacitance of the circuit?

(*b*) What is the charge stored on each capacitor?

(*c*) What is the total energy stored in the system?

Ans. (*a*) 11.5 μF; (*b*) 15.6 μC, 30 μC, 82 μC, 11 μC; (*c*) 8.28×10^{-4} J

Problem 4.72. Four capacitors are connected as in Fig. 4-21 and a voltage of 12 V is connected across the circuit. The capacitances are 1.3 μF, 2.5 μF, 6.8 μF and 0.92 μF.

(*a*) What is the equivalent capacitance of the circuit?

(*b*) What is the charge stored on each capacitor?

(*c*) What is the total energy stored in the system?

Ans. (*a*) 2.55 μF; (*b*) 10.5 μC, 20.1 μC, 26.9 μC, 3.6 μC; (*c*) 1.84×10^{-4} J

Problem 4.73. A capacitor filled with air has a capacitance of 25 μF. What capacitance would the capacitor have if it were filled with paper?

Ans. 82.5 μF

Problem 4.74. An air filled capacitor has a capacitance of 25 μF. If 1/4 of its volume were filled with paper, what capacitance would it have? (See Problem 4.46.)

Ans. 30.3 μF

Problem 4.75. An air filled capacitor has a capacitance of 25 μF, and a constant voltage of 18 V is across the capacitor.

(*a*) How much charge is stored on this capacitor?

(*b*) If the capacitor were filled with paper, and the voltage remained the same, how much charge would be stored on the capacitor?

(*c*) How much energy is stored in the system in each case?

Ans. (*a*) 4.5×10^{-4} C; (*b*) 1.49×10^{-3} C; (*c*) 4.05×10^{-3} J, 1.34×10^{-2} J

Problem 4.76. A capacitor has an area of 91 m^2 and the plates are separated by 0.86 mm. We want the capacitor to have a capacitance of 25 μF. What must be the dielectric constant of the material filling the capacitor to give this capacitance?

Ans. 26.7

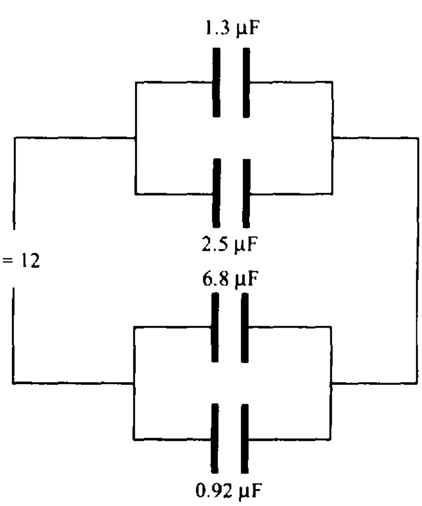

Fig. 4-21