Chapter 9

Rigid Bodies I: Equilibrium and Center of Gravity

In Sections 4.2 and 4.3, we discussed the concepts of translational and rotational equilibrium, as well as the general requirement for translational equilibrium for particles and rigid bodies. The requirements for rotational equilibrium of rigid bodies were also discussed for two simple cases: that of a body acted on by only two forces and that of a body acted on by three forces. It would be useful to review those two sections now. To deal with the general case of equilibrium of rigid bodies, when an arbitrary number of forces are acting, we must use the concept of *torque*, or *moment*.

9.1 THE TORQUE OR MOMENT OF A FORCE

Definitions

The words *torque* and *moment* are synonymous, and we will use them interchangeably. In this chapter, as in Chapter 4, we deal almost exclusively with situations in which the forces acting on a rigid body are coplanar (in the same plane), allowing an algebraic definition of torque rather than the more general vector one. An important feature of the definition of the moment of a force is that it depends on the choice of a particular point in the plane relative to which the moment is defined. This will become clear from the definition.

In Fig. 9-1 we have a typical, rigid body and have displayed one of the coplanar forces **F** acting on it. We pick some arbitrary point A and define Γ_A, the moment of the force **F** about the point A, as follows: First, draw the line of action through the force **F** (represented by the dotted line in Fig. 9-1. Next, draw the perpendicular line from the point A to that line of action (represented by the dashed line of length d_A in Fig. 9-1). Then, by definition

$$\Gamma_A = \pm d_A F \tag{9.1}$$

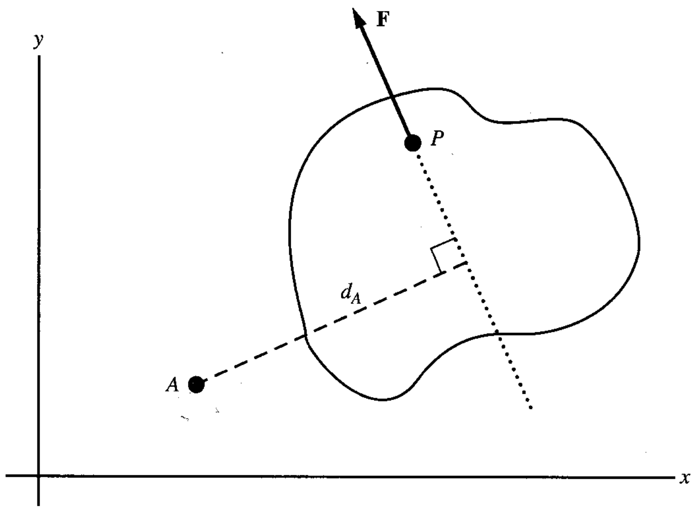


Fig. 9-1

where the choice of sign is determined by which way the force **F** would tend to rotate the body about A: clockwise or counterclockwise. (If point A is outside the body, as in Fig. 9-1, imagine that it is rigidly linked to the body and has a pin through it about which the whole system can rotate.) If point A were pinned, the force **F** would tend to rotate the body counterclockwise about the point. It is usual for such a counterclockwise moment to be considered positive, while clockwise moments are considered negative.

The distance d_A from point A to the line of action of **F** is given a special name. It is called the **moment arm** of the force **F** about A.

If more than one force is acting on a body, the total moment about A is the algebraic sum of the individual moments about A. In Fig. 9-2 we show three coplanar forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ acting on a rigid body. The total torque is given as $\Gamma_A = d_{A1}F_1 - d_{A2}F_2 + d_{A3}F_3$, where the signs have been chosen as explained above.

Note that the units of torque are force times length, the same as the units of work, although torque has a quite different physical meaning. Typical units are N · m, dyn · cm, and lb · ft.

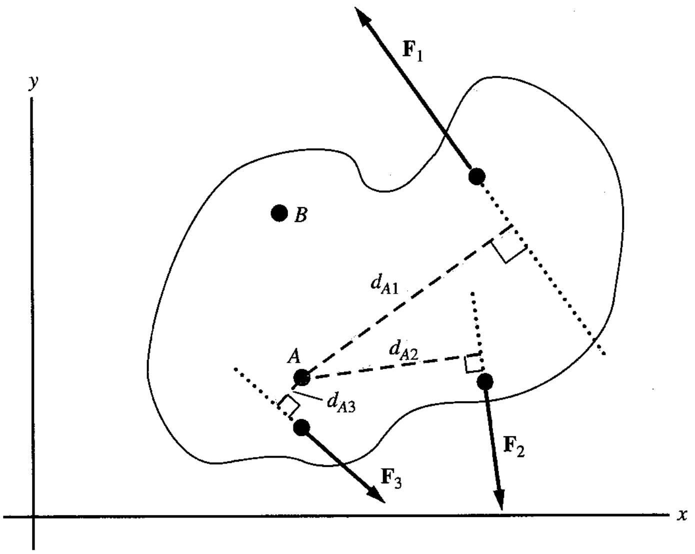


Fig. 9-2

Problem 9.1.

(*a*) Show that the torque Γ_A of the force **F** shown in Fig. 9-1 would not change if one slides **F** to a different location along its line of action.

(*b*) Show that if the force **F** were replaced by $-\mathbf{F}$ acting anywhere along the same line of action, the magnitude of the torque remains the same but the sign changes.

Solution

(*a*) From the definition, the torque depends only on the magnitude of the force and the moment arm to the line of action. Since **F** and d_A are unchanged, from Eq. (*9.1*) we see that Γ_A is unchanged.

(*b*) The magnitude of $-\mathbf{F}$ is the same as that of **F**, and d_A is unchanged, so the magnitude of Γ_A is unchanged. Now, however, the force $-\mathbf{F}$ tends to rotate the object about A in the direction opposite to that of the force **F**, so the sign of the torque must change.

Problem 9.2.

(*a*) Find the sign of the torques of $\mathbf{F}_1$ and $\mathbf{F}_2$ about A as shown in Fig. 9-3.

(*b*) What is the value of the torque of $\mathbf{F}_3$ about A?

(*c*) What are the signs of the torques of each of the three forces about point B?

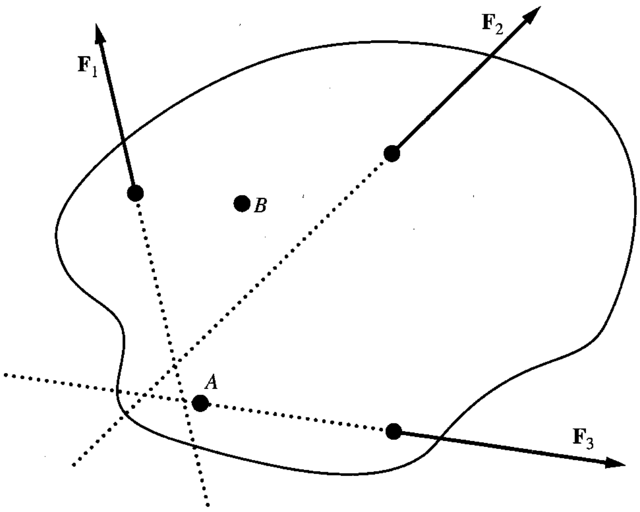


Fig. 9-3

Solution

(*a*) $\mathbf{F}_1$ has a line of action that passes very close to point A, as shown. This might make it less obvious which way $\mathbf{F}_1$ tends to rotate the object about A. To help visualize the situation we use the results of Problem 9.1, which allow us to slide $\mathbf{F}_1$ along its line of action until it is as close to point A as possible. Then it becomes obvious that $\mathbf{F}_1$ tends to rotate the object clockwise about A. Similarly, $\mathbf{F}_2$ also tends to rotate the object clockwise about A. Recall that a clockwise rotation has a negative sign.

(*b*) The line of action of $\mathbf{F}_3$ passes right through A. Then the moment arm $d_{A3} = 0$, and from Eq. (*9.1*) $\Gamma_{A3} = 0$. Thus, any force whose line of action passes through a given point has zero torque about that point.

(*c*) For point B it is not hard to see that $\mathbf{F}_2$ and $\mathbf{F}_3$ have counterclockwise moments (+), while $\mathbf{F}_1$ has a clockwise moment (−).

Problem 9.3. In Fig. 9-4 we have a force $\mathbf{F}$ acting at a given point in the body. Let $\mathbf{r}_A$ represent the relative displacement from point A to the point of application of $\mathbf{F}$. If $F = 15$ N, $r_A = 3.0$ m, and $\theta = 30°$, find the moment of $\mathbf{F}$ about point A.

Solution

Clearly $\mathbf{F}$ gives rise to a counterclockwise moment about A. To find the magnitude of the moment we must find the moment arm d_A. From the triangle we can see that $d_A = r_A \sin\theta$, so

$$\Gamma_A = d_A F = r_A F \sin\theta = (3.0\text{ m})(15\text{ N})(0.50) = 22.5\text{ N}\cdot\text{m}$$

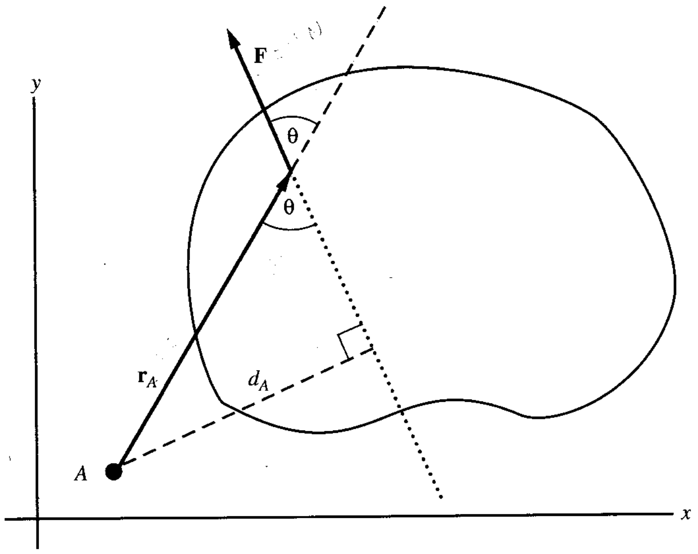


Fig. 9-4

Another View of Torque

Problem 9.3 gives us a way of expressing the torque in terms of $\mathbf{r}_A$, the displacement from an arbitrary point A to the point of application of the force **F** (Fig. 9-4). The torque due to **F** about A is

$$\Gamma_A = \pm r_A F \sin\theta \tag{9.2}$$

where θ is the angle between the vectors $\mathbf{r}_A$ and **F** when their two tails are together. This leads us to yet another expression for the torque. In Fig. 9-5 we reproduce Fig. 9-4 but now break **F** into components F_r, parallel, and F_t, perpendicular, to $\mathbf{r}_A$. As can be seen from Fig. 9-5, $F_t = F\sin\theta$. Thus, the torque from Eq. (*9.2*) can be reexpressed as $\pm r_A F_t$. We thus have three equivalent expressions for the torque:

$$\Gamma_A = \pm r_A F\sin\theta = \pm d_A F = \pm r_A F_t \tag{9.3}$$

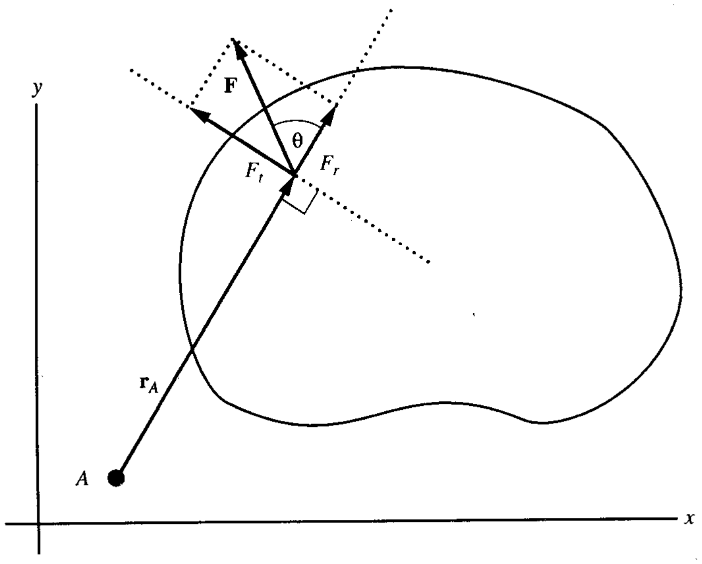


Fig. 9-5

Note the symmetry between the last two expressions of Eq. (*9.3*): F_t is the component of $\mathbf{F}$ perpendicular to $\mathbf{r}_A$, and d_A (as can be seen in Fig. 9-4) is the component of $\mathbf{r}_A$ perpendicular to $\mathbf{F}$.

Problem 9.4. Figure 9-6(*a*) shows a ladder leaning against a smooth wall, with the various forces acting on the ladder drawn in. Find the moment of the force F about point A by (*a*) finding the moment arm d_A and using Eq. (*9.1*), and (*b*) finding the component of $\mathbf{F}$ perpendicular to $\mathbf{r}_A$ and using the right side of Eq. (*9.3*).

Solution

(*a*) d_A is just the perpendicular distance from A to the line of action of $\mathbf{F}$ and is shown in Fig. 9-6(*b*). Thus, $d_A = (20\text{ m}) \sin 37° = 12$ m. From Eq. (*9.1*): $\Gamma_A = (12\text{ m})(30\text{ N}) = 360\text{ N}\cdot\text{m}$.

(*b*) Here we note that $\mathbf{r}_A$ is along the ladder from A to the contact point with the wall. Then, $F_t = F \sin 37° = (30\text{ N}) \sin 37° = 18$ N, and from Eq. (*9.3*); $\Gamma_A = r_A F_t = (20\text{ m})(18\text{ N}) = 360\text{ N}\cdot\text{m}$.

9.2 THE LAWS OF EQUILIBRIUM FOR RIGID BODIES

Translational and Rotational Equilibrium

We are now able to express the necessary and sufficient conditions for translational and rotational equilibrium of a rigid body acted on by any number of coplanar forces. Two conditions must hold:

1. The vector sum of the forces must vanish.
2. The algebraic sum of the torques about a given point must vanish.

Mathematically:

$$(1)\quad \sum \mathbf{F}_i = 0 \qquad (2)\quad \sum \Gamma_{Ai} = 0 \qquad (9.4a, b)$$

The first condition (*4a*) can be expressed in component form:

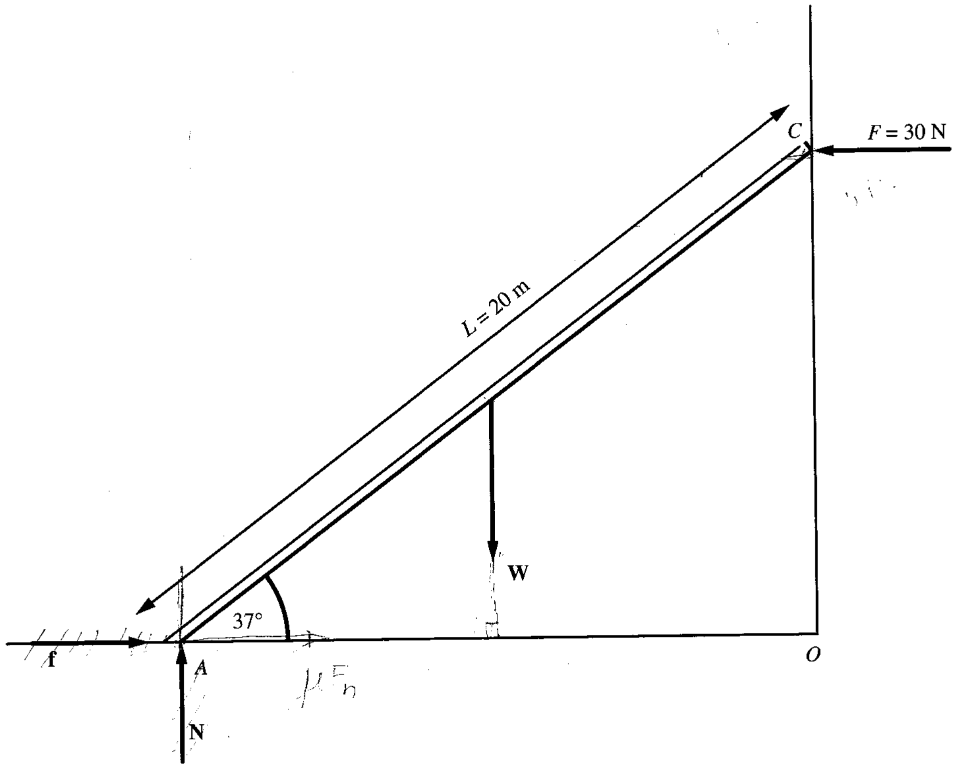
$$\sum F_{ix} = 0 \qquad \sum F_{iy} = 0 \qquad (9.5a, b)$$

Note. Equation (*9.4a*) [or Eqs. (*9.5*)] is the statement of Newton's first law applied to a particle. From our discussion of the center of mass of a system of particles in Chap. 8 these same equations are statements of the law of translational equilibrium for the CM of a rigid body. Equation (*9.4b*) can also be derived from Newton's laws for particles, as will be demonstrated in the next chapter.

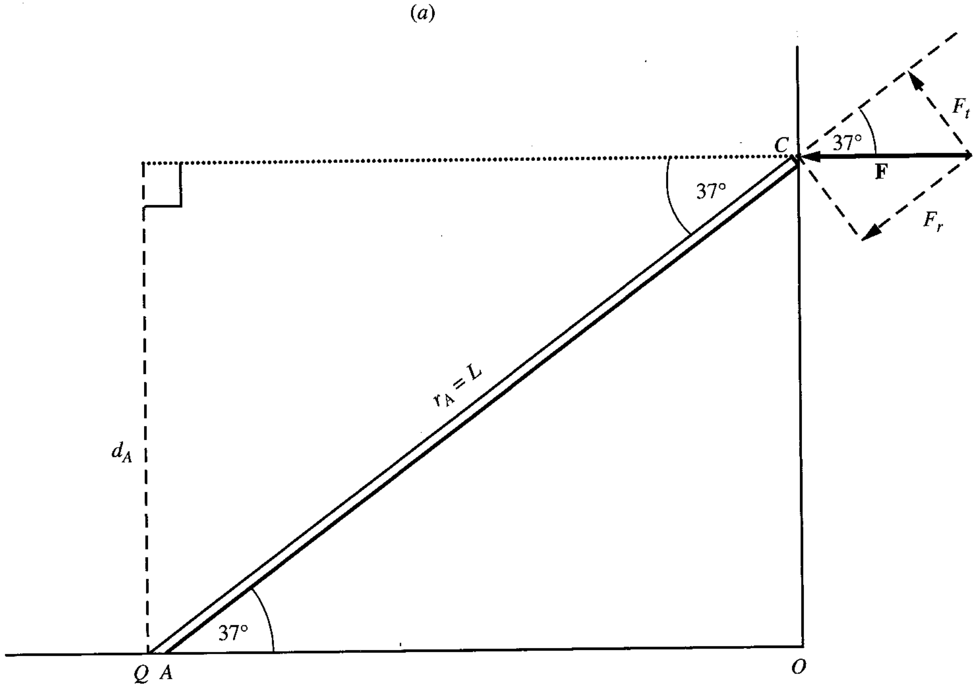
Problem 9.5. Suppose the ladder in Fig. 9-6(*a*) is at rest under the action of the forces shown. The ladder is uniform so that we can consider its weight $\mathbf{W}$ to act at its center. Calculate the values of the weight W, the normal force N, and the frictional force f.

Solution

Since the ladder is at rest, it is in both translational and rotational equilibrium. Therefore, Eqs. (*9.4*) must hold. If we take moments about A, the friction and normal forces don't contribute since their lines of action pass through A. Thus the only two forces contributing are F and W. The moment of F was



(a)



(b)

Fig. 9-6

already calculated in Problem 9.4: $\Gamma_{AF} = 360\ \text{N}\cdot\text{m}$. We note that the moment is clockwise for W, and hence negative, and that the moment arm is just half the floor distance from A to the wall:

$$d_{AW} = \tfrac{1}{2}(20\ \text{m})\cos 37^\circ = 8.0\ \text{m}$$

Then $\Gamma_{AW} = -d_{AW}W = -(8.0\ \text{m})W$, and Eq. (*9.4b*) yields

$$\Gamma_{AF} + \Gamma_{AW} = 360\ \text{N}\cdot\text{m} - (8.0\ \text{m})W = 0 \qquad \text{or} \qquad W = 45\ \text{N}$$

From the first condition of equilibrium, Eqs. (*9.5*), we have, in the horizontal and vertical directions, respectively,

$$F - f = 30\ \text{N} - f = 0 \qquad \text{or} \qquad f = 30\ \text{N}$$
$$N - W = N - 45\ \text{N} = 0 \qquad \text{or} \qquad N = 45\ \text{N}$$

Problem 9.6. In Fig. 9-7 we have a light rod of length 5.0 ft free to pivot about a horizontal axis through point A. A weight of 10 lb hangs from one end.

(*a*) What force F must be applied at the other end to keep the rod from rotating?

(*b*) What then is the force N exerted by the pivot on the rod?

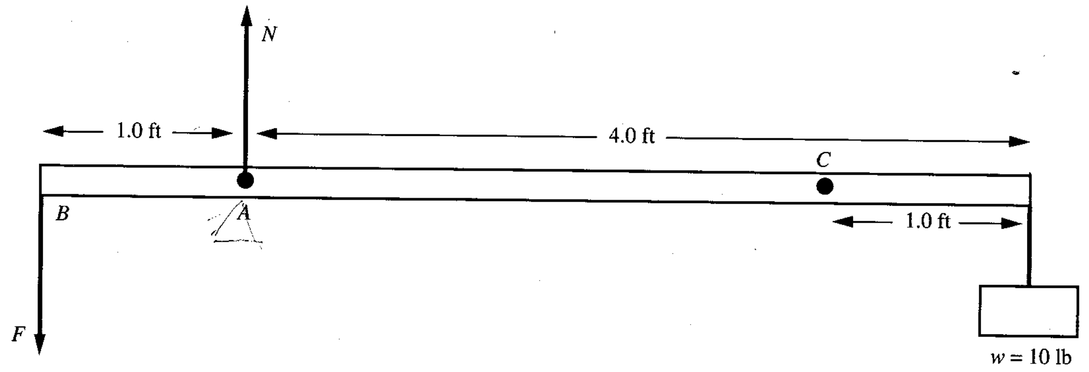


Fig. 9-7

Solution

(*a*) Since the rod is light, we may neglect its weight. By taking moments about A, we get

$$(1.0\ \text{ft})F - (4.0\ \text{ft})(10\ \text{lb}) = 0 \qquad \text{or} \qquad F = 40\ \text{lb}$$

(*b*) For translational equilibrium Eqs. (*9.5*) must be obeyed. Taking the y components we have

$$N - F - w = N - 40\ \text{lb} - 10\ \text{lb} = 0 \qquad N = 50\ \text{lb}$$

Choice of Points about Which to Take Moments

In Problems 9.5 and 9.6 we made specific choices of the point about which moments were to be taken for use in Eq. (*9.4b*). Would we have gotten different results if we had taken moments about some other point? Is there a "correct" point about which to take moments in applying the laws of equilibrium?

The following result can be proved: If the first condition of equilibrium, Eq. (*9.4a*), holds, and if Eq. (*9.4b*) holds about a particular point A, then Eq. (*9.4b*) will also hold about every other point as well. Thus, once the vector sum of forces adds up to zero, the sum of the moments about one point will be zero if, and only if, the sum of the moments about every other point is zero.

This result is particularly useful for two reasons. One is that you can always check your solution to an equilibrium problem by taking moments about a different point and checking that they add up to zero. The other is that you have complete flexibility in picking the point about which to calculate moments for Eq. (*9.4b*). This means that you can pick the point that is most convenient for solving the problem. For example, in Problem 9.5, point A is a particularly convenient choice since it eliminates the forces **f** and **N** from the moment equation and hence leaves just one unknown, the weight W.

Problem 9.7. Check your answer to Problem 9.5 by taking moments about point C in Fig. 9-6(a).

Solution

The sum of the moments about point C must add up to zero. The forces, as determined in Problem 9.5 are $F = 30$ N, $W = 45$ N, $f = 30$ N, $N = 45$ N. Referring to Fig. 9.6(a), we note that F has no moment about C; W and f have counterclockwise (positive) moments about C; N has a clockwise (negative) moment about C. The moment arms are easily obtained by looking at perpendicular distances from C to the lines of action of the forces: $d_{CW} = \frac{1}{2}(20\text{ m})\cos 37° = 8.0$ m; $d_{CN} = (20\text{ m})\cos 37° = 16$ m; $d_{Cf} = (20\text{ m})\sin 37° = 12$ m. Then

$$\Gamma_C = d_{CW}W + d_{Cf}f - d_{CN}N = (8.0\text{ m})(45\text{ N}) + (12\text{ m})(30\text{ N}) - (16\text{ m})(45\text{ N})$$
$$= 360\text{ N}\cdot\text{m} + 360\text{ N}\cdot\text{m} - 720\text{ N}\cdot\text{m} = 0$$

Note. In getting the moments of the forces acting on the ladder at point A we treated **f** and **N** as separate forces even though they are acting at the same point in the body. We could have replaced them by their vector sum and considered the moment of that single force. There was no point in doing so since the moment arms to the individual forces **f** and **N** are easily obtained, while getting the moment arm to the vector sum of **f** and **N** would have been harder. Actually, the reverse process is often more useful: If one is given a single force acting at a point, it is sometimes easier to break the force into a sum of two forces whose moment arms are easier to calculate.

Problem 9.8. Check the results of Problem 9.6 (Fig. 9-7) by taking moments (a) about point B at the left end of the rod, and (b) about point C, which is 1 ft from the right end of the rod.

Solution

(a) The force F contributes zero moment about the left end of the rod. Then

$$\Gamma_B = (1.0\text{ ft})N - (5.0\text{ ft})w = (1.0\text{ ft})(50\text{ lb}) - (5.0\text{ ft})(10\text{ lb}) = 50\text{ lb}\cdot\text{ft} - 50\text{ lb}\cdot\text{ft} = 0$$

(b) About point C all three forces contribute:

$$\Gamma_C = (4.0\text{ ft})F - (3.0\text{ ft})N - (1.0\text{ ft})w$$
$$= (4.0\text{ ft})(40\text{ lb}) - (3.0\text{ ft})(50\text{ lb}) - (1.0\text{ ft})(10\text{ lb})$$
$$= 160\text{ lb}\cdot\text{ft} - 150\text{ lb}\cdot\text{ft} - 10\text{ lb}\cdot\text{ft} = 0$$

Problem 9.9. A uniform rectangular block of weight $w = 40$ N moves at a constant speed along a frictionless horizontal surface, under the action of forces $\mathbf{F}_1$ and $\mathbf{F}_2$, as shown in Fig. 9-8.

(a) Find the magnitude of the force $\mathbf{F}_2$ and of the normal force **N**.

(b) Find the point of application of the normal force, as measured from the left end of the block.

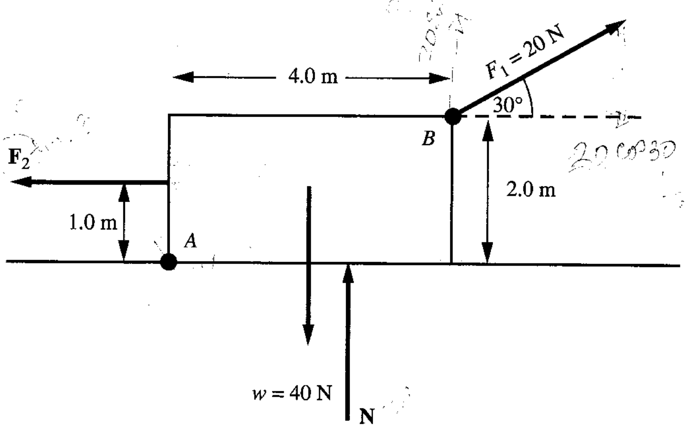


Fig. 9-8

Solution

(*a*) Since the block is undergoing translation at constant velocity and is not rotating, it is in equilibrium. We first consider the first condition of equilibrium, Eqs. (*9.5*). For the x direction

$$F_1 \cos 30° - F_2 = 0 \qquad \text{or} \qquad F_2 = 17.3 \text{ N}$$

For the y direction

$$F_1 \sin 30° + N - 40 \text{ N} = 0 \qquad \text{or} \qquad N = 30 \text{ N}$$

Thus, part (*a*) is solved without resort to the second condition of equilibrium.

(*b*) Here we need to find the location of the line of action of the normal force. Clearly only a moment equation will give us such information. We take moments about the point A at the left lower corner of the block. Since it will be difficult to find the distance of the line of action of the moment arm of $\mathbf{F}_1$ from A, we replace $\mathbf{F}_1$ by a pair of forces, one horizontal and one vertical, corresponding to the components F_{1x} and F_{1y}. These forces have magnitudes $F_{1x} = 17.3$ N and $F_{1y} = 10$ N. Then

$$\Gamma_A = (1.0 \text{ m})F_2 - (2.0 \text{ m})w + (4.0 \text{ m})F_{1y} - (2.0 \text{ m})F_{1x} + xN = 0$$

Substituting we get

$$\Gamma_A = (1.0 \text{ m})(17.3 \text{ N}) - (2.0 \text{ m})(40 \text{ N}) + (4.0 \text{ m})(10 \text{ N}) - (2.0 \text{ m})(17.3 \text{ N}) + x(30 \text{ N}) = 0$$

or

$$x = 1.91 \text{ m}$$

Problem 9.10. Check the results of Problem 9.9 by taking moments about point B.

Solution

The force $\mathbf{F}_1$ does not contribute a moment about point B, so

$$\Gamma_B = (2.0 \text{ m})w - (1.0 \text{ m})F_2 - (4.0 \text{ m} - x)N = 0$$
$$= (2.0 \text{ m})(40 \text{ N}) - (1.0 \text{ m})(17.3 \text{ N}) - (2.09 \text{ m})(30 \text{ N}) = 80 - 17.3 - 62.7 = 0$$

Problem 9.11. A weight $W = 500$ N hangs from one end of a uniform horizontal beam of weight $w = 100$ N and length L whose other end is pivoted at point A on the wall [Fig. 9-9(*a*)]. The beam is supported by a wire making an angle of 40° with the beam. Find (*a*) the tension in the wire, (*b*) the vertical and horizontal components of the force exerted by the pivot on the beam.

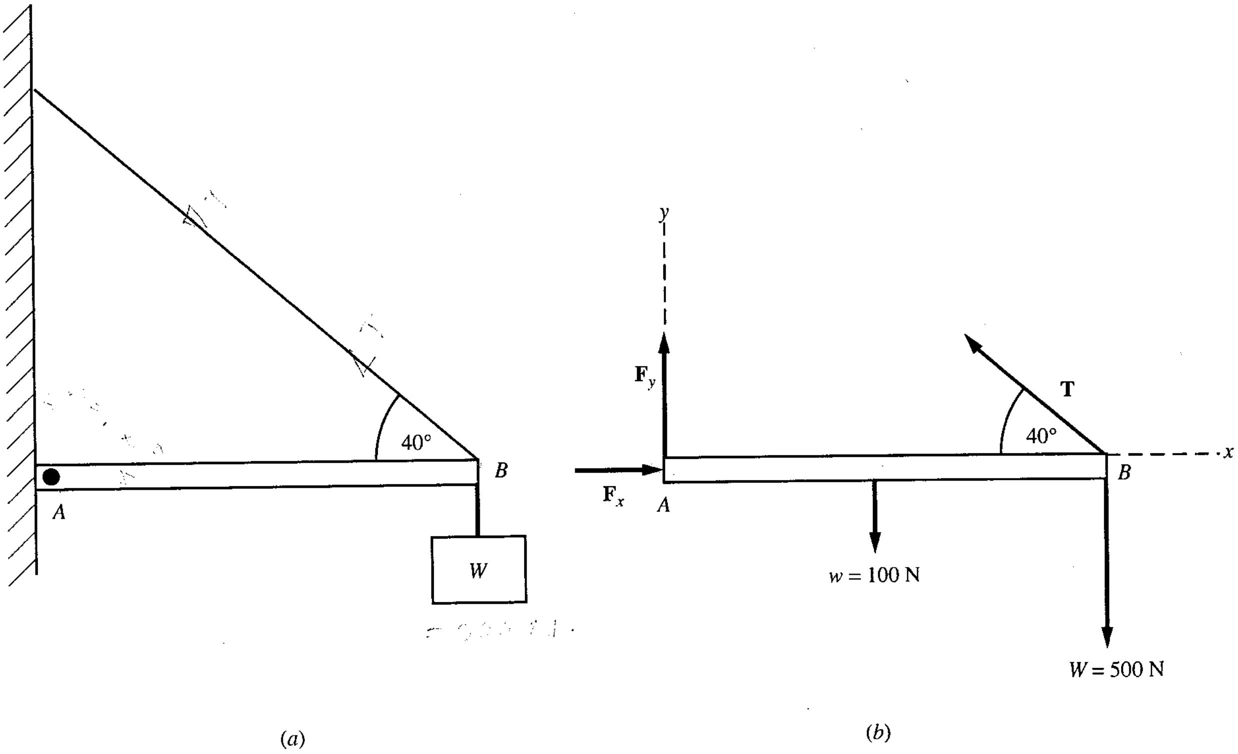


Fig. 9-9

Solution

(*a*) The forces acting on the beam are shown in Fig. 9-9(*b*). Since the beam is in equilibrium, Eqs. (*9.4b*) and (*9.5a, b*) must hold. If we take moments about point B, the forces **T**, **W**, and $\mathbf{F}_x$ do not contribute since their lines of action pass through B. The only unknown force in the equation will then be F_y, so we can solve for it.

$$\Gamma_B = \left(\frac{L}{2}\right)w - LF_y = 0 \qquad \text{or} \qquad \tfrac{1}{2}(100\text{ N}) - F_y = 0 \qquad \text{or} \qquad F_y = 50\text{ N}$$

To find T we use Eq. (*9.5b*):

$$F_y - w - W + T\sin 40^\circ = 0 \qquad \text{or} \qquad 50\text{ N} - 100\text{ N} - 500\text{ N} + (0.643)T = 0$$

$$\text{or} \qquad T = 855\text{ N}$$

(*b*) We already obtained F_y = 50 N. To get F_x we use Eq. (*9.5a*)

$$F_x - T\cos 40^\circ = 0 \qquad \text{or} \qquad F_x = (855\text{ N})(0.766) = 655\text{ N}$$

Problem 9.12. If the beam in Problem 9.11 could be considered weightless, what would then be the values of T, F_y, and F_x?

Solution

We could redo the formal steps of Problem 9.11 with w set equal to zero, but it is easier to approach the problem more directly. First we note that if $w = 0$, the only force contributing any moment about point B is F_y. Since the sum of the moments must be zero, we have $F_y = 0$. Then, from translational equilibrium,

$$T_y - W = 0 \qquad \text{or} \qquad (0.643)T = 500 \text{ N} \Rightarrow T = 778 \text{ N}$$

Finally from Eq. (9.5a)

$$T_x - F_x = 0 \qquad \text{or} \qquad F_x = (0.766)(778 \text{ N}) = 596 \text{ N}$$

Problem 9.13. The uniform boom of length L (Fig. 9-10) weighs $w = 800$ N and supports a load of $W = 1000$ N on one end.

(*a*) Find the tension T in the support wire.

(*b*) Find the magnitude and direction of the force **F** exerted on the boom by the pivot at point A.

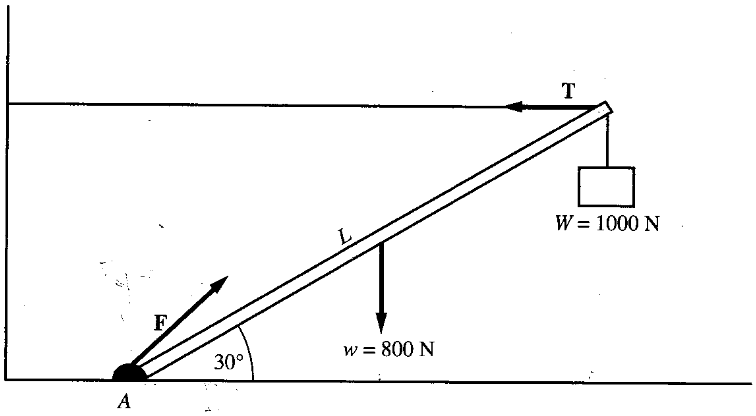


Fig. 9-10

Solution

(*a*) We take moments about point A, which eliminates the force **F** from the moment equation. Then $\Gamma_A = (L \sin 30°)T - (L \cos 30°)W - \frac{1}{2}(L \cos 30°)w = 0$. Dividing out L we get

$$(0.50)T - (0.866)(1000 \text{ N}) - (0.433)(800 \text{ N}) = 0 \qquad \text{or} \qquad T = 2425 \text{ N}$$

(*b*) Breaking **F** into x and y components, we get from Eq. (9.5a):

$$F_x - T = 0 \qquad \text{or} \qquad F_x = 2425 \text{ N}$$

from Eq. (9.5b):

$$F_y - w - W = 0 \qquad \text{or} \qquad F_y = 1800 \text{ N}$$

Then

$$F = [(2425 \text{ N})^2 + (1800 \text{ N})^2]^{1/2} = 3020 \text{ N}$$

If θ is the angle of **F** above the positive x axis,

$$\tan\theta = \frac{F_y}{F_x} = \frac{1800}{2425} = 0.742 \qquad \text{or} \qquad \theta = 36.6°$$

Problem 9.14. A uniform ladder of length $L = 60$ ft and weight $w = 50$ lb leans against a frictionless wall. The ladder makes an angle of 50° with the floor, and the coefficient of friction between ladder and floor is $\mu_s = 0.50$. A painter, weighing (with bucket) $W = 160$ lb, starts to climb the ladder. What distance x along the ladder can the painter go before the ladder starts to slip?

Solution

The situation is depicted in Fig. 9-11, with all the forces on the ladder drawn in. So long as the ladder is in translational equilibrium

$$N = w + W = 210 \text{ lb} \qquad \text{and} \qquad f = F$$

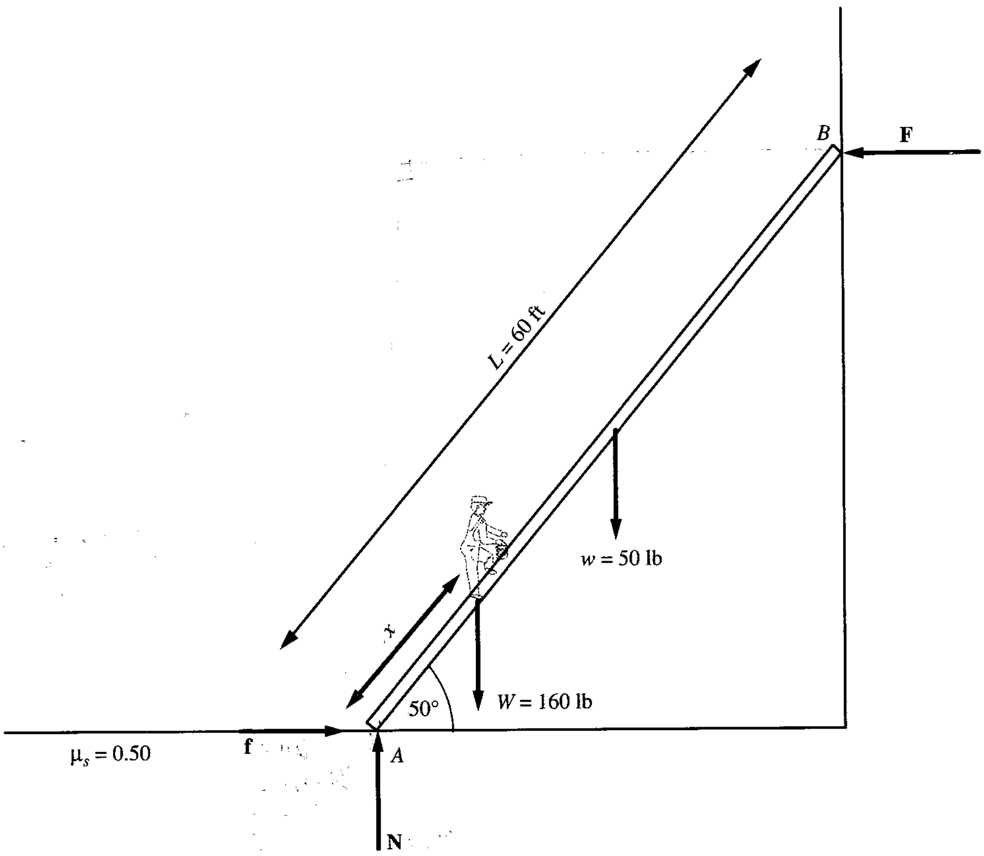


Fig. 9-11

independent of the position of the painter. In the moment equation (9.4b), about point A, only F appears as an unknown; furthermore, as the painter climbs the ladder, the clockwise torque about A due to **W** is increasing, because the moment arm is increasing. This can be balanced only by an increasing counterclockwise torque due to **F**. Since the moment arm from A to the line of action of **F** is fixed, the torque due to **F** can increase only if F increases. Since $f = F$, this means that f must keep increasing as the painter climbs the ladder, until it reaches its maximum value,

$$f_{\max} = \mu_s N = (0.50)(210 \text{ lb}) = 105 \text{ lb} = F_{\max}$$

Thus, just before slipping takes place, (9.4) gives

$$\begin{aligned}\Gamma_A &= [(60 \text{ ft})\sin 50°]F_{\max} - [(30 \text{ ft})\cos 50°]w - [x_{\max}\cos 50°]W \\ &= (60 \text{ ft})(0.766)(105 \text{ lb}) - (30 \text{ ft})(0.643)(50 \text{ lb}) - x_{\max}(0.643)(160 \text{ lb}) = 0\end{aligned}$$

whence $x_{\max} = 37.5$ ft.

9.3 EQUIVALENT SETS OF COPLANAR FORCES

We have seen that to have translational and rotational equilibrium for a rigid body acted on by a set of coplanar forces, the set of forces must obey precisely two conditions, as expressed by Eqs. (*9.4a, b*). Nothing further need be known about the details of the set of forces to ensure equilibrium. The body will still be in equilibrium if our original set of forces is replaced by any other set of forces obeying the same equations. This leads us to an intriguing question. Suppose a rigid body acted on by a set of coplanar forces is *not* in equilibrium and we want to fully describe its translational and rotational motion. What do we need to know about the set of forces to completely describe the body's motion?

Our discussion of center of mass in Chap. 8 indicated that the acceleration of the CM is completely determined by the resultant external force acting on the body. In Chap. 10 we will see that the rotational motion of the body depends only on the resultant external torque acting on the body. From these results we can deduce that if we have two different sets of coplanar forces, and each set adds up to the same resultant force and gives rise to the same resultant torque (about any given point), then each of the sets will affect the motion of the object in precisely the same way. Let us state without proof that:

One can always replace one set of forces acting on a rigid body by any other set of forces having the same vector sum and the same resultant torque (about any chosen point) to get the same effect on the motion of the body.

Center of Gravity

The above result turns out to be extremely useful. It explains why we are justified in assuming that the weight of a rigid body is a single force acting at a particular point in the body, even though there are myriad forces due to gravity on the individual molecules making up the body. It also explains why we can assume that the normal force acting on the surface of one object by another can be assumed to be a single force acting at a given point on the object, even though, in reality, there are myriad forces acting between the molecules of the two surfaces. Indeed it can be shown that, except for one special case (discussed below), any set of coplanar forces acting on a rigid body can be replaced by a *single force* (their resultant), acting along a particular line of action.

Problem 9.15. Find the single force **F** that can replace the two parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the body shown in Fig. 9-12, and find its point of application.

Solution

The single replacement force must equal the vector sum of the original forces. Since the forces are parallel, the force **F** points in the same direction and has magnitude $F = F_1 + F_2 = 30\text{ N} + 20\text{ N} = 50\text{ N}$. Further, the moment of **F** about the origin must equal the combined moments of the original two forces: $\Gamma = \Gamma_1 + \Gamma_2 \Rightarrow xF = x_1F_1 + x_2F_2$

$$x(50\text{ N}) = (6.0\text{ m})(30\text{ N}) + (12.0\text{ m})(20\text{ N}) \qquad \text{or} \qquad x = 8.40\text{ m}$$

The force **F** can act anywhere along its line of action.

Problem 9.16. Find the single force that can replace the three forces shown in Fig. 9-13, and find its line of action.

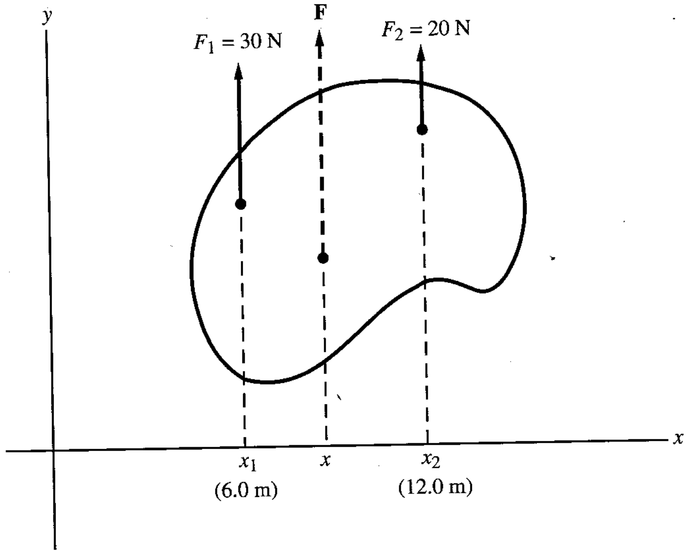


Fig. 9-12

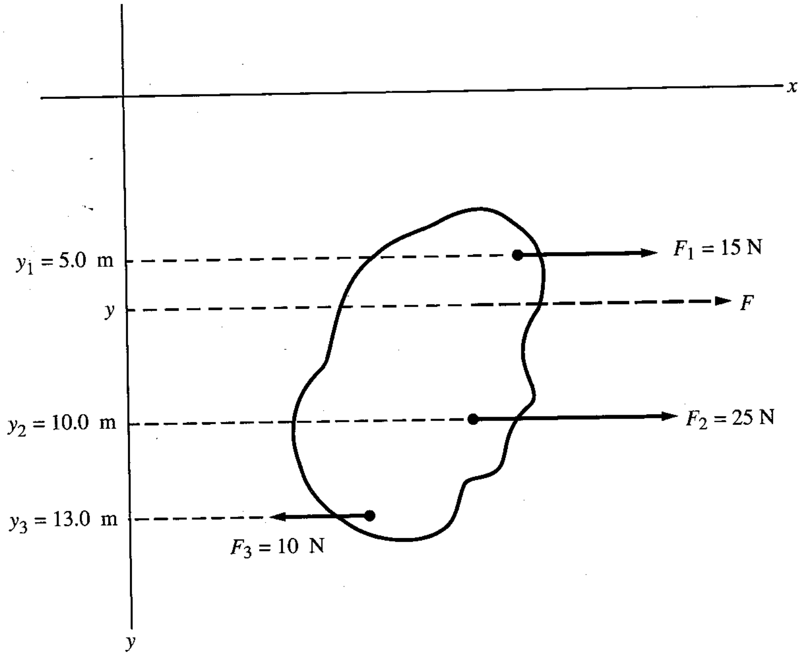


Fig. 9-13

Solution

Again, our single force is parallel to $\mathbf{F}_1$ and $\mathbf{F}_2$ and has magnitude

$$F = F_1 + F_2 - F_3 = 15 \text{ N} + 25 \text{ N} - 10 \text{ N} = 30 \text{ N}$$

The line of action is determined by equating moments about the origin:

$$yF = y_1F_1 + y_2F_2 - y_3F_3$$

Substituting, we get

$$y(30 \text{ N}) = (5.0 \text{ m})(15 \text{ N}) + (10 \text{ m})(25 \text{ N}) - (13 \text{ m})(10 \text{ N}) \qquad \text{or} \qquad y = 6.5 \text{ m}$$

Problem 9.17. Figure 9-14 depicts a flat irregular plate in the xy plane, where y is the vertical direction upward from the earth's surface. The weight vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_i \ldots$, represent the pull of gravity on the various individual molecules of the plate. Show that (*a*) the single force $\mathbf{W}$ which can replace these forces is just the weight of the object; (*b*) the line of action of $\mathbf{W}$ passes through the CM of the plate.

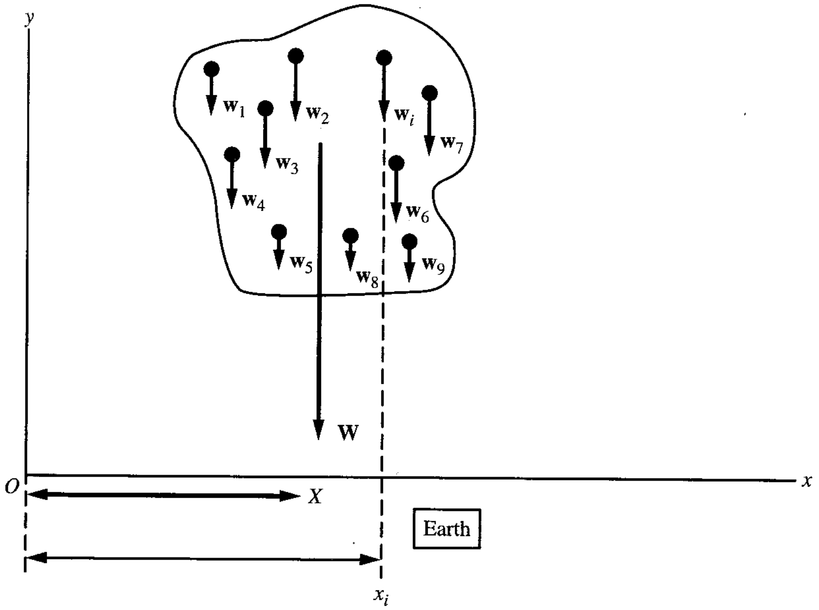


Fig. 9-14

Solution

(*a*) Since all the individual **w**'s are parallel, their resultant **W** must point in the same direction and must be of magnitude

$$W = w_1 + w_2 + \cdots + w_i + \cdots = \Sigma w_i \qquad (i)$$

where W equals the total weight of the plate.

(*b*) We equate moments about the origin. Let x_i be the moment arm of $\mathbf{w}_i$ and X the moment arm of **W**. Then, $XW = \Sigma x_i w_i$. Dividing by W, using Eq. (*i*), and recalling that $w_i = m_i g$, we get

$$X = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{\Sigma m_i x_i}{M} \qquad (ii)$$

where M is the total mass of the plate. Comparing Eq. (*ii*) to Eq. (*8.19a*), we see that $X = X_{CM}$.

The results of Problem 9.17 can be extended to any arbitrary orientation of the plate, in three dimensions as well as in two dimensions. For example, by considering the object to be rotated by 90° in the x, y plane and redoing the problem we find that the single force **W** which replaces all the individual weights is still the total weight of the object, and its line of action of **W** still passes through the center of mass. Thus, no matter what the orientation, one can always assume the weight of a rigid body acts at the CM. The point in a body where the total weight can be assumed to be acting is often called the **center of gravity** (CG). Thus the center of gravity and the center of mass are one and the same point.

Problem 9.18. Locate the CG of the composite object, made of three copper strips shown in Fig. 9-15.

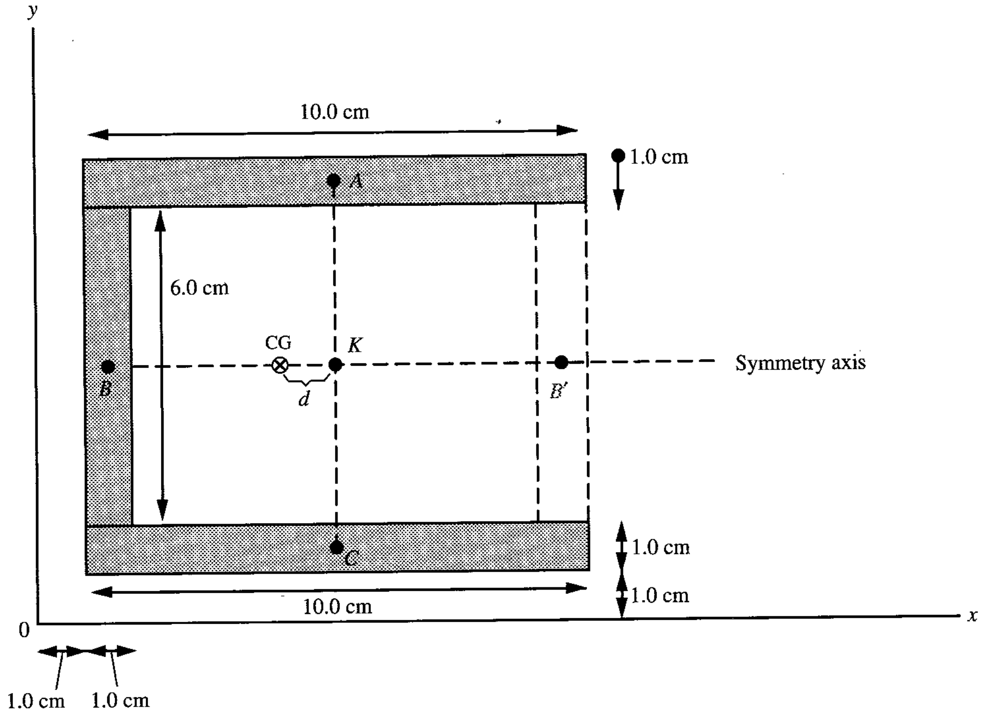


Fig. 9-15

Solution

The CM or CG of each strip lies at its geometric center. By symmetry we also know that the overall CG of the composite lies somewhere along a horizontal line through the CG of strip B. The total weight of the object passes through this overall CG, and it must have a moment about the origin equal to the sum of the moments of the weights of the individual strips:

$$X_{CG}W = x_A w_A + x_B w_B + x_C w_C$$

and

$$X_{CG} = \frac{x_A w_A + x_B w_B + x_C w_C}{w_A + w_B + w_C} \qquad (i)$$

From Fig. 9-15, we have $x_A = x_C = 6.0$ cm and $x_B = 1.5$ cm. While we are not given the weights of the strips, we do know they are all made of the same material, so their weights are proportional to their areas, and $w_B = 0.60w_A$; $w_C = w_A$. Substituting this into (*i*) and dividing out w_A, we get $X_{CG} = (x_A + 0.60x_B + x_C)/(1.0 + 0.60 + 1.0) = [6.0\ \text{cm} + 0.60(1.5\ \text{cm}) + 6.0\ \text{cm}]/2.6 = 4.96\ \text{cm}$. Thus the overall CG is 3.46 cm to the right of the CG of strip B.

Problem 9.19. A ladder consists of two wood segments of equal length and a crosspiece of negligible weight, as shown in Fig. 9-16. When the ladder is open, both segments make an angle of 60° with the floor. Each segment is uniform but of different weight, as shown. Find the CG of the open ladder.

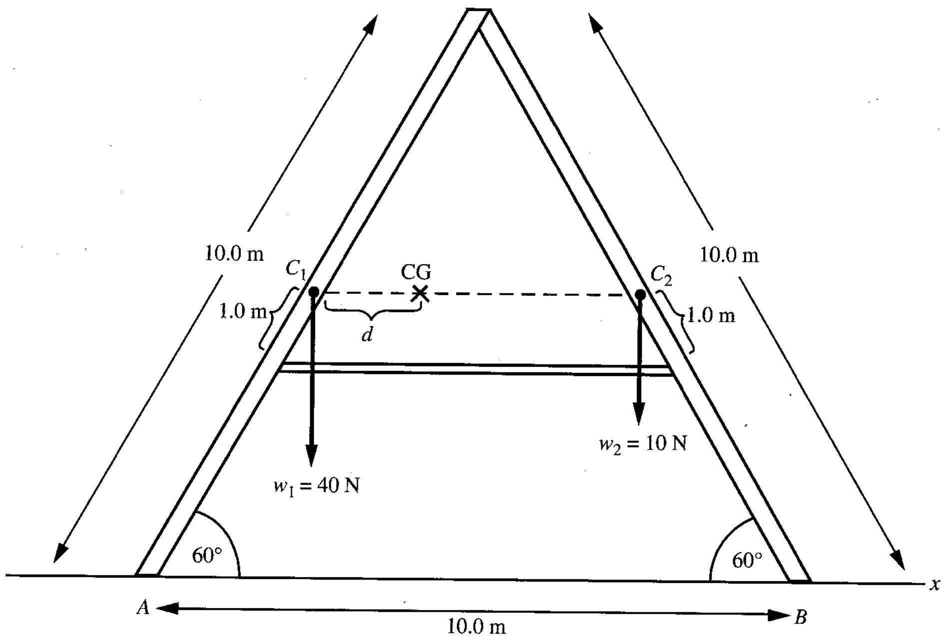


Fig. 9-16

Solution

The CGs of the two segments are at the same height above the ground. If the earth were tilted by 90° and pulling in the direction of the x axis, the weights of both segments would have a common line of action. The overall weight would, of course, have to have the same line of action. Thus we know that the overall CG must lie along the horizontal line between the two individual CGs. To find where along the line it acts, we go back to the actual situation with the earth pulling downward. Then, taking moments about point A, we get

$$X_{\mathrm{CG}} = \frac{x_1 w_1 + x_2 w_2}{w_1 + w_2} \qquad (i)$$

where x_1 and x_2 are the moment arms to the weights w_1 and w_2, of left and right segments, respectively. We have $x_1 = (5.0\text{ m})\cos 60° = 2.5$ m; $x_2 = (10\text{ m})\cos 60° + (5.0\text{ m})\cos 60° = 7.5$ m. Substituting into Eq. (i), we get

$$X_{\mathrm{CG}} = \frac{(2.5\text{ m})(40\text{ N}) + (7.5\text{ m})(10\text{ N})}{50\text{ N}} = 3.5\text{ m}$$

Couples

We return to the exceptional case in which a set of coplanar forces cannot be replaced by a single force. As long as the resultant of the set of coplanar forces is not zero, we can always find a point of application, far or near as needed, so that the torque of the resultant matches the torque of the set itself. But, what happens if the resultant force is zero, while the resultant torque is not zero? Then the zero resultant force can never give rise to the needed torque! Figure 9-17(a) depicts such a situation. Clearly the resultant of the three forces acting on the body is $F = F_1 + F_2 - W = 20\text{ lb} + 10\text{ lb} - 30\text{ lb} = 0$. The sum of the torques about the origin is

$$\Gamma = (5.0\text{ ft})(20\text{ lb}) + (15\text{ ft})(10\text{ lb}) - (10\text{ ft})(30\text{ lb}) = -50\text{ lb}\cdot\text{ft}$$

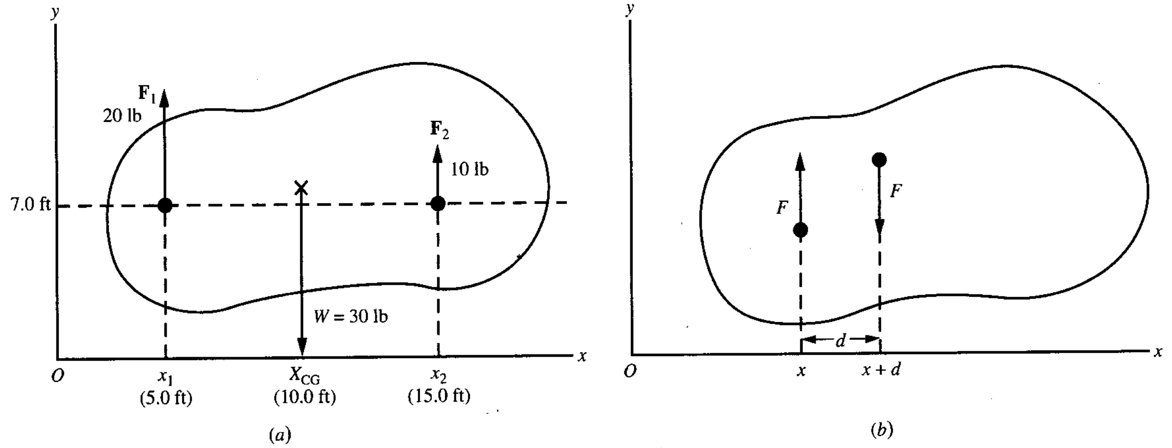


Fig. 9-17

For such situations, we can't replace the system by a single force, but we can replace the system by a pair of equal and opposite forces displaced a distance d from each other. Figure 9-17(b) shows such a pair of forces for the case at hand. Clearly the pair of forces sums to zero. In addition the torque due to the pair is $xF - (x + d)F = -dF$. For any choice of x one gets the same torque, so the location of the pair is unimportant. Furthermore, one has complete flexibility in the choice of F, as long as the product dF gives the desired result. A pair of equal and opposite forces giving rise to a torque is called a **couple**. In picking the couple for this example we chose the upward force to the left of the downward force to assure that the torque about the origin came out negative. As we saw earlier, the torque due to a couple does not depend on the absolute location of the couple, but rather only on the choice of d and F.

In Fig. 9-18 we depict three objects that are tilted slightly from their equilibrium positions on a horizontal table: a cylinder (a) and a cone (b) with broad support bases initially touching the table surface, and an inverted cone (c) with just the apex touching the table surface. In each case the objects

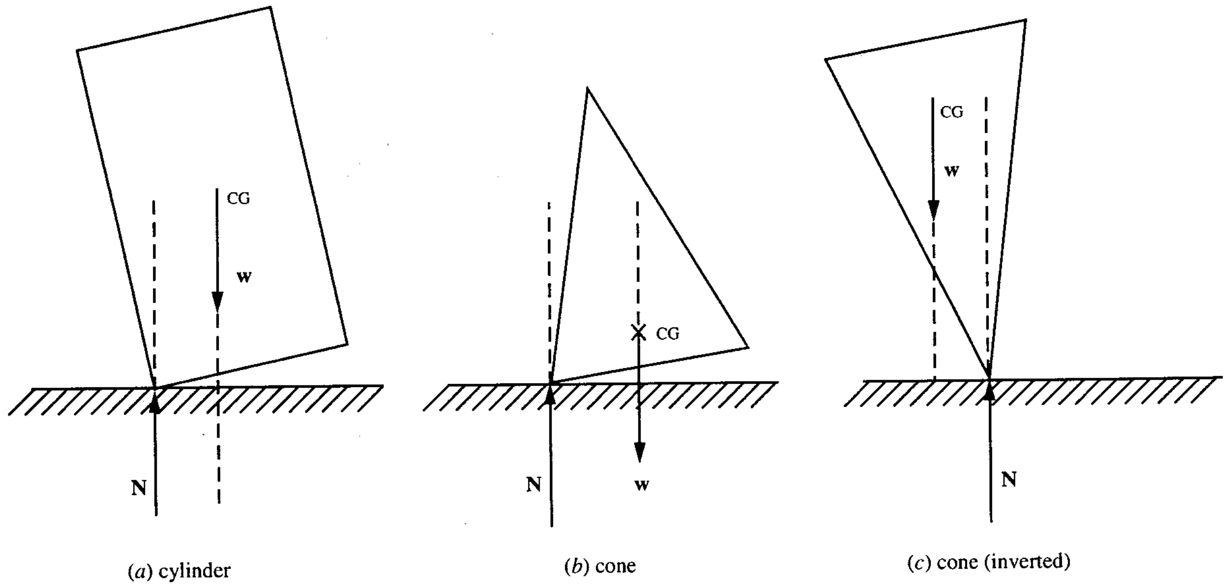


Fig. 9-18

are acted on by two forces, the weight downward acting at the CG (or CM) and a normal force upward. In equilibrium these are equal and opposite. Furthermore, before tilting, they also have a common line of action. That is, the normal force appears directly under the CG to assure equilibrium. After a slight tilt to the left, for our first two cases, the normal force acts on the leftmost edge of the broad base, a point which is to the left of the CG. We have, in each case, a couple that gives a clockwise moment and tends to return the object to its equilibrium position. Similarly, had the tilt been to the right, we would have a counterclockwise couple that again tends to return the object to its equilibrium position. For a situation in which every slight tilt from equilibrium gives rise to a couple that restores equilibrium, we say that the equilibrium is **stable.**

Case (*c*) is quite different. Here, a slight tilt either way gives rise to a moment that tends to make the cone tilt even more in the same direction, so the cone falls over on its side. Whenever a slight tilt of an object away from equilibrium gives rise to a couple that continues the motion away from equilibrium, we say the equilibrium is *unstable.*

Problem 9.20. Determine the kind of equilibrium we have for the three objects in Fig. 9-19.

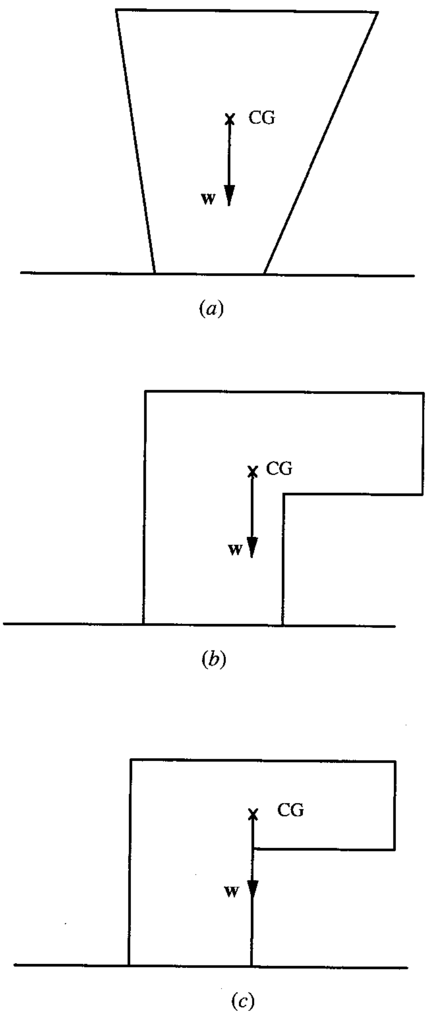


Fig. 9-19

Solution

(*a*) In the equilibrium position, the CG is between the two edges of the base touching the ground. A slight tilt to the left will therefore put the normal force under the left edge, while the CG will still be to the right of that point; the couple will therefore restore the object to the equilibrium position. The same reasoning holds for a slight tilt in the other direction. Thus we have stable equilibrium.

(*b*) The CG lies to the left of the right edge on the ground. A slight tilt to the right will leave it so, and we get a couple that restores equilibrium. The result is even more obvious for a tilt to the left. We thus again have stable equilibrium.

(*c*) The CG is directly over the right edge on the ground. Now even the slightest tilt to the right will put the CG to the right of the normal force, and the object will topple. We thus have unstable equilibrium.

Problem 9.21. What kind of equilibrium do we have for the two objects shown in Fig. 9-20, a uniform cylinder and a uniform cone lying on their sides?

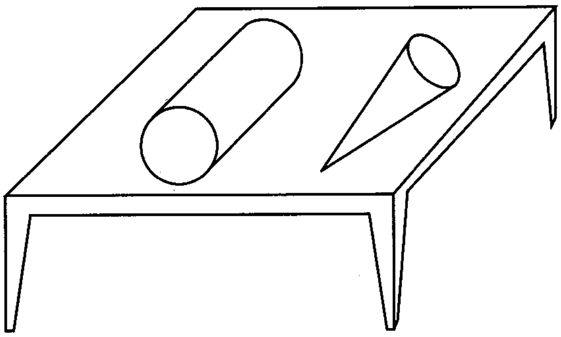


Fig. 9-20

Solution

A slight motion of either object leaves the normal force directly below the CG. Thus if one moves the cylinder by rolling it slightly to a new position and then releasing it from rest, it will stay in equilibrium in the new position. It will neither return to its original position nor move further away. It is thus in neither stable nor unstable equilibrium. It is said to be in **neutral equilibrium**.

Problem 9.22.

(*a*) Two identical heavy lead weights are suspended from the ends of a light, thin, bent aluminum rod, that is balanced at its center on a pivot. [Fig. 9-21(*a*)]. Describe the nature of the equilibrium of the system.

(*b*) A plastic horse and rider are connected rigidly by a stiff curved wire to a heavy iron ball, as shown in Fig. 9-21(*b*). The horse is supported by one foot on a narrow platform. Describe what happens when the horse and rider are tilted in any direction.

Solution

(*a*) In the equilibrium position the CG is located directly below the pivot point midway along and slightly above the line between the centers of the two weights. Although the CG is located outside

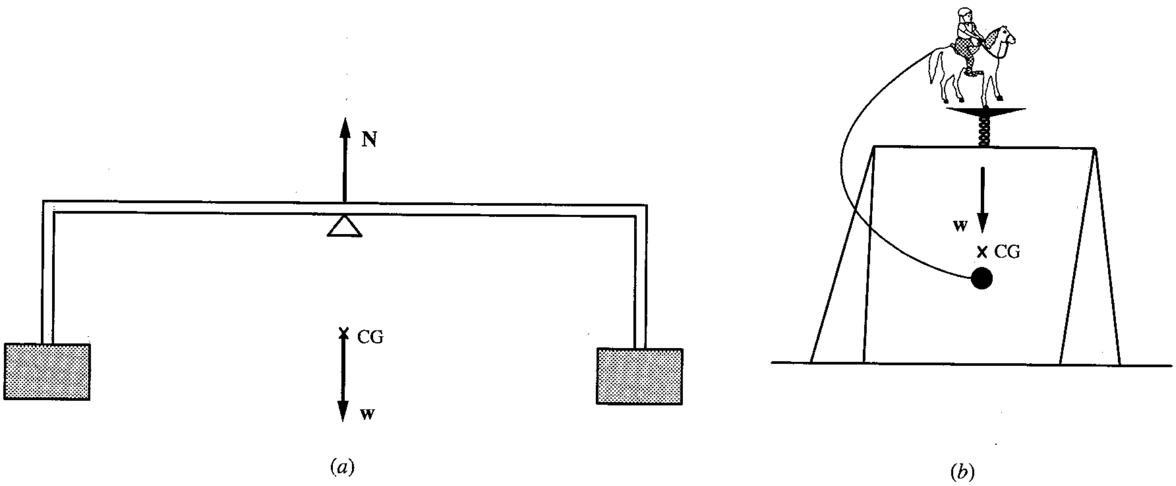


Fig. 9-21

the body, it maintains a geometrically fixed position relative to the body as the body moves. When the system is not touched, it is acted on by only two forces: gravity at the CG and the upward normal force at the pivot. At equilibrium these forces are equal and opposite and have a common line of action. When the body is rotated slightly in the counterclockwise direction about the pivot, the CG moves to the right, and the two forces form a clockwise couple, which returns the object to the equilibrium position when the object is let go. Similarly, after a clockwise rotation, the object will again return to the equilibrium position. The equilibrium is therefore stable. A slight tap on either side will set up oscillations about the equilibrium position because the object picks up speed as it returns to equilibrium and overshoots the equilibrium position, where the process is reversed. If friction is low, these oscillations can last a long time.

(*b*) The situation here is essentially the same as in part (*a*) because the heavy ball lowers the CG to below the pivot, as shown. Tilting the horse in any direction and letting go leads to oscillations about the equilibrium position.

Problems for Review and Mind Stretching

Problem 9.23. Show that the statement of Chap. 4, that for a body to be in equilibrium under the action of three forces the forces must be concurrent, follows from Eqs. (*9.4a, b*).

Solution

From Eq. (*9.4a*) we have that the vector sum of the three forces adds up to zero, so the three vectors form a triangle and therefore are in the same plane. For them to be concurrent, their lines of action must pass through a common point. Consider the point of intersection of the lines of action of two of the forces, and call it point A. In taking moments about A, these two forces don't contribute. Equation (*9.4b*) then implies that the third force must have zero moment about A, so its line of action passes through A as well.

Problem 9.24. A uniform horizontal rod 2.0 m long and weighing $w = 20$ N has weights of 80 and 40 N hanging from its ends (Fig. 9-22). Find (*a*) the magnitude and direction of the fourth force **F** necessary to keep the rod in equilibrium; (*b*) the point of application of the force F on the rod.

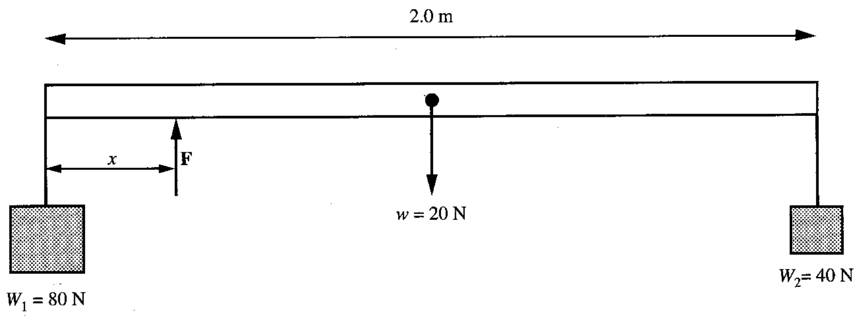


Fig. 9-22

Solution

(*a*) From the first condition of equilibrium, Eq. (*9.4a*), we must have that **F** is vertically upward and balances the other three forces:

$$F = 80\ \text{N} + 20\ \text{N} + 40\ \text{N} = 140\ \text{N}$$

(*b*) From the second condition of equilibrium, Eq. (*9.4b*),

$$x(140\ \text{N}) - (0\ \text{m})(80\ \text{N}) - (1.0\ \text{m})(20\ \text{N}) - (2.0\ \text{m})(40\ \text{N}) = 0 \qquad \text{or} \qquad x = 0.714\ \text{m}$$

Problem 9.25. A large wooden crate 8.0 ft high, 3.5 ft wide, and of weight $w = 100$ lb rests on a horizontal surface with coefficient of static friction $\mu_s = 0.60$. Assume the CG of the crate is at its geometric center. A horizontal force F is applied to the crate to get it moving. Below what height h must be force F be applied if the crate is to start to slide before it starts to tip over?

Solution

The situation is depicted in Fig. 9-23. The crate is acted on by the four forces: **w**, **N**, **f**, and **F**, where **f** is the retarding frictional force and **N** is the normal force of the floor on the crate. Let x represent the

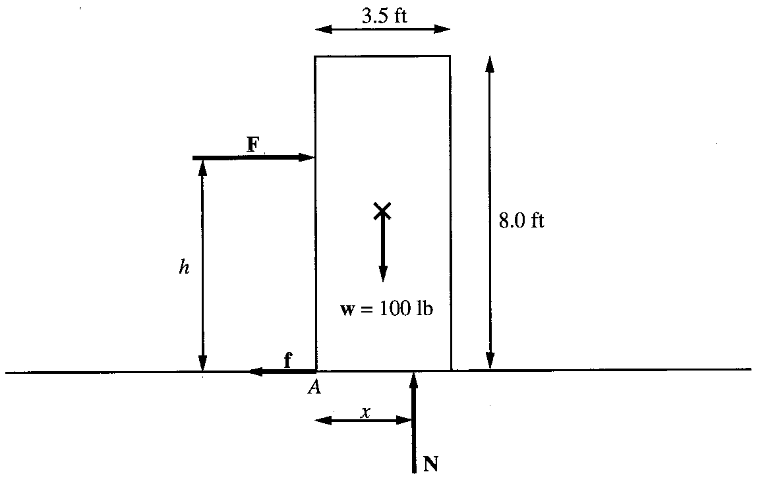


Fig. 9-23

distance from the left edge of the crate, point A, to the point of application of $\mathbf{N}$. To just get the crate to slide, we must have

$$F = f_{max} = \mu_s N \qquad (i)$$

From equilibrium in the vertical direction, $N = w = 100$ lb. Then (*i*) gives $F = 0.60(100 \text{ lb}) = 60$ lb. The question of whether the crate tips or not can be resolved by examining where the normal force acts. If **N** acts at the extreme right edge, the entire crate loses contact with the floor, except at that edge; this is the condition for just starting to tip. Taking moments about A, we see that **f** does not contribute, while **w** contributes a fixed clockwise moment. For fixed $F = 60$ lb, the clockwise moment of **F** increases with the height h at which it is applied. To balance the clockwise moments of **F** and **w**, we have only the counterclockwise moment due to **N**. For fixed $N = 100$ lb, this moment can get larger only by increasing the moment arm x. We therefore set x equal to its maximum possible value, 3.5 ft, to determine h_{max}:

$$\Gamma_A = (3.5 \text{ ft})(100 \text{ lb}) - h_{max}(60 \text{ lb}) - (1.75 \text{ ft})(100 \text{ lb}) = 0 \qquad \text{or} \qquad h_{max} = 2.92 \text{ ft}$$

Problem 9.26. A uniform door, of height 3.50 m and width 1.50 m and weighing 200 N, is supported by two small hinges, as shown in Fig. 9-24. The hinges are symmetrically placed, 20 cm from the top and bottom of the door. If the top hinge supports the full weight of the door, find the horizontal forces exerted by each hinge on the door.

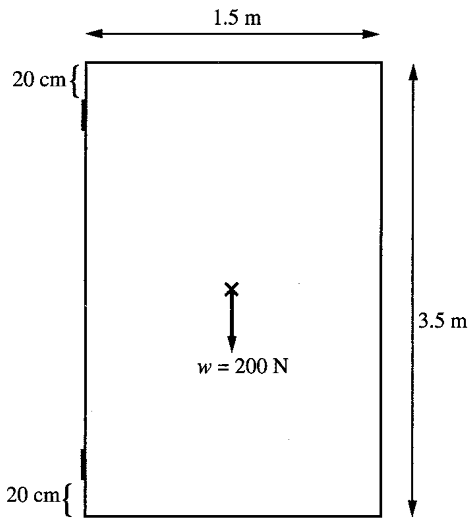


Fig. 9-24

Solution

Since the only horizontal forces on the door are the hinge forces, they must be equal and opposite; let their common magnitude be F. The top hinge also exerts a vertical force of 200 N on the door to balance the weight of the door. If we calculate moments about the lower hinge, the only forces contributing are the weight $w = 200$ N and the horizontal force F due to the upper hinge. Since the weight gives a clockwise moment, force F must be to the left. Then

$$\Gamma = (3.50 \text{ m} - 0.40 \text{ m})F - (0.75 \text{ m})(200 \text{ N}) = 0 \qquad \text{or} \qquad F = 48.4 \text{ N}$$

Problem 9.27. Find the CG of the uniform disk of radius $R = 2.0$ m with a small disk of radius $r = R/3$, cut out, as shown in Fig 9-25.

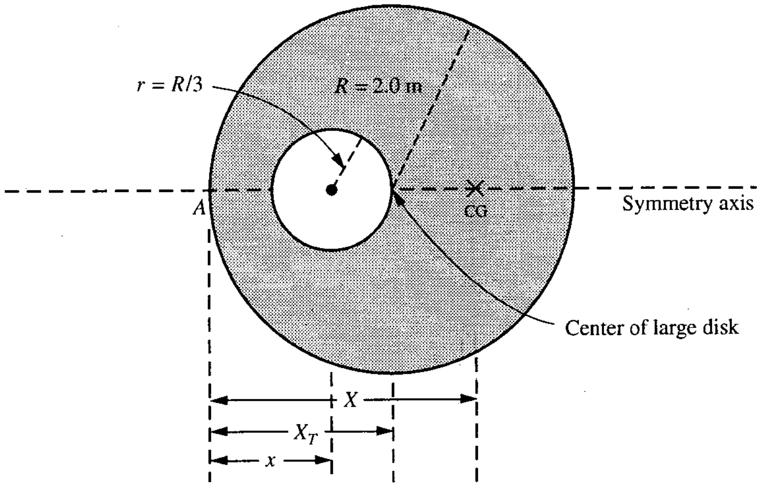


Fig. 9-25

Solution

At first this seems to be a formidable problem, but consider the disk with the piece missing. We know, by symmetry, that the CG (or CM) lies along the x axis, as shown, but we don't know where along the axis. If we put the cutout piece back in place, the CG of the *combination* would be just that of the whole disk and would be at its center. Assume that X is the moment arm from point A to the CG of the disk with the piece missing, and W is its weight. Similarly let x be the moment arm to the CG of the cutout piece and w be its weight. Then let X_T be the CG for the combined (complete) disk. We have

$$x = R - r = R - \frac{R}{3} = 1.33 \text{ m} \qquad \text{and} \qquad X_T = R = 2.0 \text{ m} \tag{i}$$

Also, the weights of the two pieces are proportional to their areas:

$$w = \sigma\pi r^2 \qquad W = \sigma\pi(R^2 - r^2) \tag{ii}$$

where σ is the proportionality constant. Note that in the second term, for W, we have subtracted out the area of the cutout. Then, solving for the CG of the combination of the two pieces, we get

$$X_T(w + W) = xw + XW \tag{iii}$$

Substituting and simplifying common terms, we get

$$X_T(R^2) = xr^2 + X(R^2 - r^2) \qquad \text{or} \qquad X_T = \frac{x}{9} + \frac{8X}{9}$$

so that $X = 2.08$ m.

Supplementary Problems

Problem 9.28. Find the sign of the torques of the three forces in Fig. 9-2 about point B. Use the standard convention.

Ans. Γ_1 is positive; Γ_2 is negative; Γ_3 is positive

Problem 9.29. Assume that the moment of the force in Fig. 9-4 about A is 100 N·m. If $d_A = 20$ m and $r_A = 60$ m, find (*a*) F; (*b*) θ.

Ans. (*a*) $F = 5.0$ N; (*b*) $\theta = 19.5°$

Problem 9.30. Assume that the rod of Fig. 9-7 is free to pivot about point A and that a 10-lb weight hangs at one end as shown. If the rod is uniform and has a weight of 3.0 lb, find the value of (*a*) the force F necessary for equilibrium; (*b*) the pivot force N necessary for equilibrium.

Ans. (*a*) 44.5 lb; (*b*) 57.5 lb.

Problem 9.31. For the boom-and-weight system of Fig. 9-26, find (*a*) the tension in the wire, (*b*) the magnitude and direction of the force exerted by the wall on the boom.

Ans. (*a*) 108 N; (*b*) 78 N, 46° above horizontal

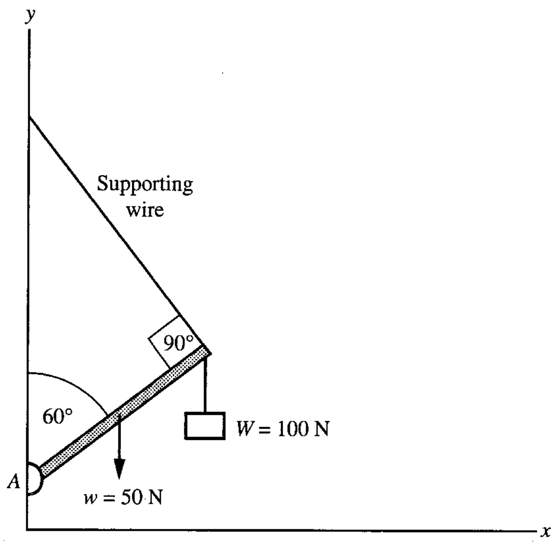


Fig. 9-26

Problem 9.32. Assume that the boom in Fig. 9-9(*a*) is weightless, that the maximum tension the wire can withstand without snapping is 3000 N, and that the maximum compressive force the boom can withstand without buckling is 2000 N.

(*a*) When the hanging weight W is increased, which will happen first: the wire snapping or the boom buckling?

(*b*) At what values of T and W will the event in part (*a*) occur?

Ans. (*a*) The boom will buckle; (*b*) $T = 2610$ N, $W = 1680$ N

Problem 9.33. How will the results of Problem 9.32 change if the uniform boom had a weight of 600 N?

Ans. (*a*) The boom will again buckle; (*b*) $T = 2610$ N, $W = 1380$ N

Problem 9.34. Find the x and y components of the force **F** exerted on the boom by the wall for (*a*) Problem 9.32(*b*), (*b*) Problem 9.33(*b*).

Ans. (*a*) $F_x = 2000$ N, $F_y = 0$ N; (*b*) $F_x = 2000$ N, $F_y = 300$ N

Problem 9.35. The ladder shown in Fig. 9-16 rests on a frictionless horizontal surface. The two segments of the ladder are hinged at the top and held together by a weightless horizontal crosspiece. Find the normal forces exerted by the floor on the ladder at points A and B.

Ans. $N_A = 32.5$ N; $N_B = 17.5$ N

Problem 9.36. Referring to Problem 9.35 and Fig. 9-16, assume that the crosspiece is attached at a point 4.0 m along the length of each ladder segment as measured from the bottom. Find the tension in the crosspiece. [*Hint*: Take one leg of the ladder as the system and apply the laws of equilibrium to it, using the results of Problem 9.35.]

Ans. 12.0 N

Problem 9.37. In Fig. 9-27 a block of weight $w = 80$ N is being pulled at constant speed along a horizontal surface by a force **F** acting at 20° above the horizontal. The coefficient of kinetic friction between surface and block is $\mu_k = 0.40$. Determine F and N, using only the first condition of equilibrium in the x and y directions.

Ans. $F = 29.7$ N; $N = 69.8$ N

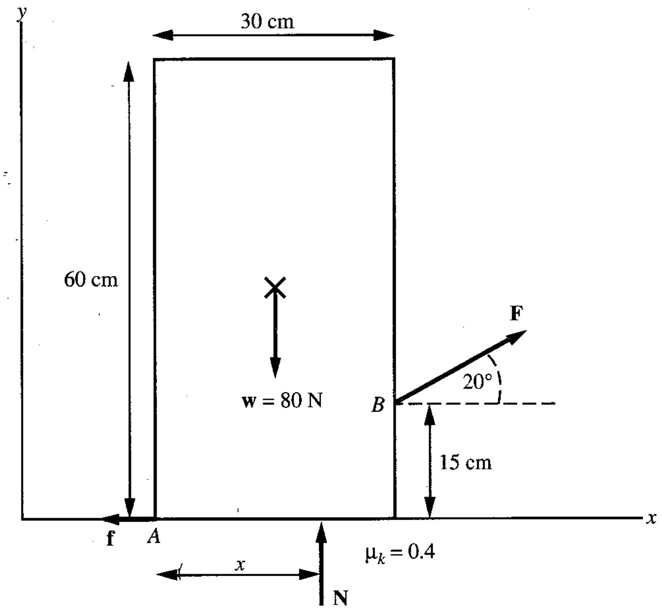


Fig. 9-27

Problem 9.38. Referring to Problem 9.37, find the distance from the left edge of the block to the point of application of the normal force.

Ans. 18.8 cm

Problem 9.39. Suppose that in Problem 9.37 the force **F** remained in the same direction but had a higher point of application B. How high could point B get before the block started to tip over?

Ans. 43 cm

Problem 9.40. For the uniform boom of length L supported by a horizontal wire (Fig. 9-28), find (*a*) the tension in the wire, (*b*) the magnitude and direction of the force exerted on the boom at the pivot.

Ans. (*a*) 8.74 kN; (*b*) 10.6 kN, 34.5° above horizontal

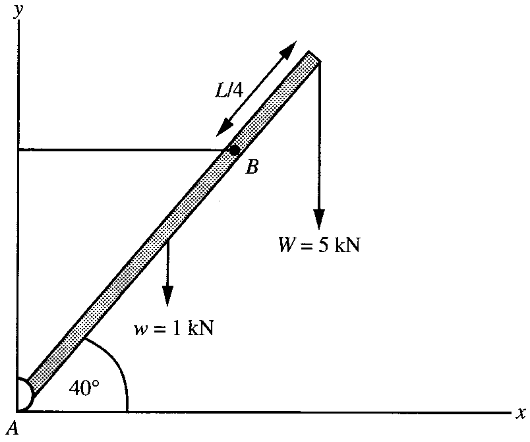


Fig. 9-28

Problem 9.41. The wire in Problem 9.40 has a breaking point of 12.0 kN. Assuming the point of attachment B of the wire is placed lower on the boom while the wire is kept horizontal, how far down along the boom can point B get before the wire snaps?

Ans. $0.454L$ from the top

Problem 9.42. A uniform seesaw of length 20 ft has two youngsters of weights w_a = 100 lb and w_b = 40 lb, sitting on the ends (Fig. 9-29). Find the proper location x of the pivot for the seesaw to be just in balance, if (*a*) the weight of the seesaw can be ignored, (*b*) the seesaw weighs 30 lb.

Ans. (*a*) 5.71 ft; (*b*) 6.47 ft

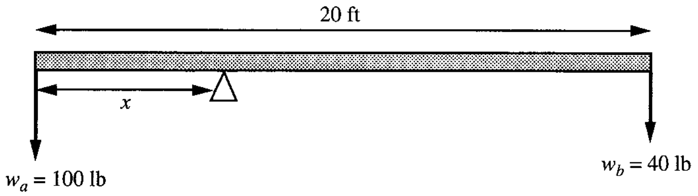


Fig. 9-29

Problem 9.43. A very thin rod bent into the shape of a right angle (90°) is made of a material which weighs 30 N per linear meter. In addition to gravity, there are two forces acting to keep the rod fixed: $\mathbf{F}_1$ at the elbow and $\mathbf{F}_2$ at point B (Fig. 9-30).

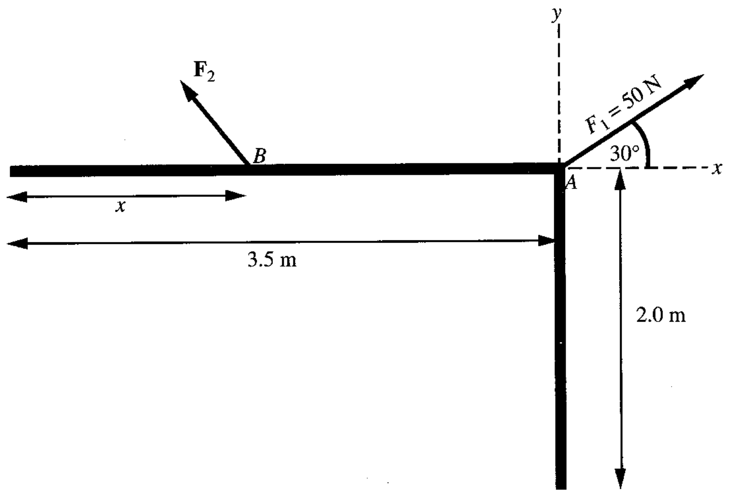


Fig. 9-30

(*a*) Find the magnitude and direction of $\mathbf{F}_2$.

(*b*) Find the distance x to the point of application of $\mathbf{F}_2$.

Ans. (*a*) 147 N, 72.8° above negative x axis; (*b*) 2.19 m

Problem 9.44. In Fig. 9-31 a rigid object, whose weight can be ignored, is acted on by the three forces shown.

(*a*) Find the x and y components of the single force $\mathbf{F}$ that can replace these three forces.

(*b*) Find the total torque of the three forces about the origin.

Ans. (*a*) $F_x = 88.5$ N, $F_y = 84.9$ N; (*b*) -15.9 N · m

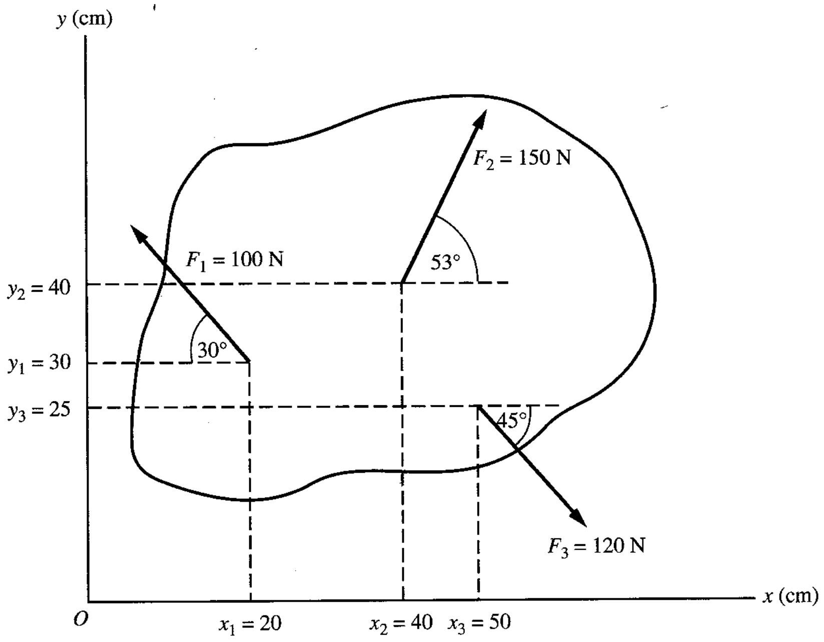


Fig. 9-31

Problem 9.45. A point on the line of action of **F** in Problem 9.44 has abscissa $x = 30$ cm. What is the corresponding ordinate? [*Hint*: Assume that the force **F** acts at this point on its line of action, and determine y from the torque requirements.]

Ans. $y = 46.7$ cm

Problem 9.46. Find the x and y coordinates of the CG of the bent rod in Fig. 9-30. Use the coordinate system shown in the figure.

Ans. $x = -1.11$ cm; $y = -0.36$ cm

Problem 9.47. Find the CG of the asymmetrical dumbbell of Fig. 9-32.

Ans. 14.7 cm left of center of large sphere, along symmetry axis

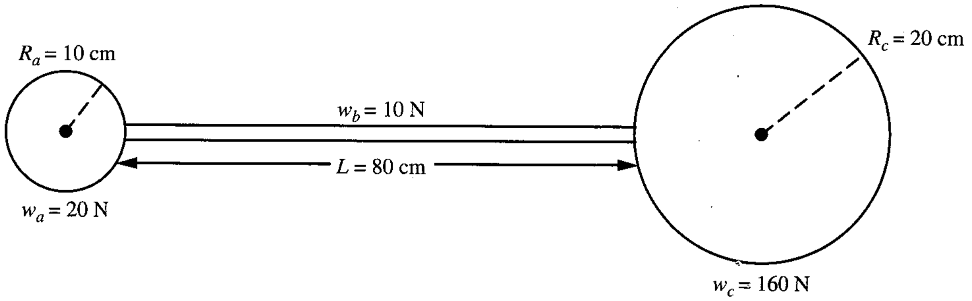


Fig. 9-32

Problem 9.48. Suppose that in addition to the weight the only other force acting on the dumbbell of Problem 9.47 were an upward force **F** of magnitude 190 N. Find the horizontal distance to the line of action of **F**, as measured from the center of the large sphere, if the resulting couple gave (*a*) a clockwise moment of 1600 N · m, (*b*) a counterclockwise moment of 5600 N · m.

Ans. (*a*) 23.1 cm to left; (*b*) 14.8 cm to right