Chapter 10

Rigid Bodies II: Rotational Motion

10.1 ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

In this section we will consider the motion of a rigid body that is free to rotate about a fixed axis. This means that all the particles of the body initially on this axis remain in fixed locations on this axis as the body moves. Our first task will be to describe the motion, or kinematics, of the body as a whole. Next we will relate the overall motion of the body to the motion of the individual particles making it up.

Description of Rotational Motion

In Fig. 10-1 we illustrate a rigid body free to rotate about a fixed axis, which we assume is perpendicular to the plane of the figure. We let the rotation axis be the z axis of our coordinate system. We can then choose x and y axes in the plane of the figure as shown. These axes are assumed to be *fixed in space in an inertial reference frame*; they do not rotate with the body. Since, by hypothesis, the particles of the body lying along the z axis stay fixed as the body rotates, every other particle in the body is constrained to move about the z axis on a circular arc with a radius equal to the perpendicular distance from the axis to the particle in question. These circular arcs are always parallel to the xy plane.

Angular Displacement

We now describe the motion of the body as a whole. We note that if we etch a series of straight-line segments parallel to the xy plane in the body, then as the body rotates, the etched lines will rotate as well (Fig. 10-2). In fact, since the body is rigid, the line segments must maintain their positions

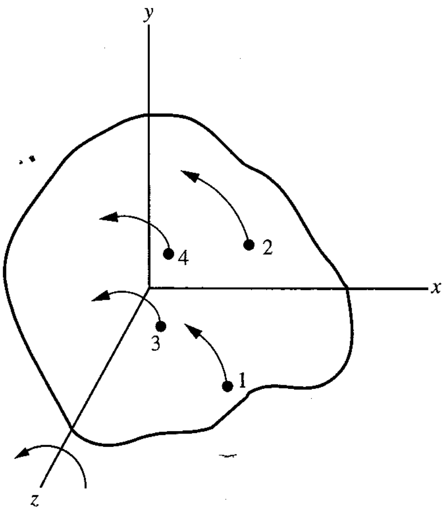

Fig. 10-1

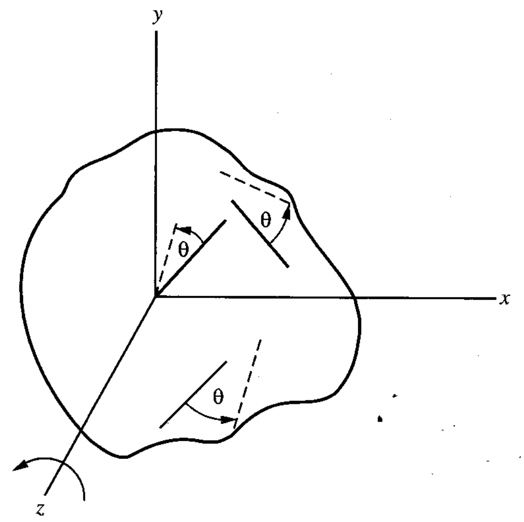

Fig. 10-2

relative to one another so that every such line segment will rotate through exactly the same angle in a given time interval. This is illustrated in Fig. 10-2, where the solid line segments correspond to one instant of time and the dashed lines correspond to the same line segments at some later instant. Thus the orientation of the rigid body can be completely specified by giving the orientation angle θ of a single chosen line segment etched in the body. For convenience we choose a segment that has one end (permanently) at the origin of our coordinate system, as shown in Fig. 10-3(a). We then let θ, measured in radians, be the angle the guideline makes with the x axis at time t. [Recall that the angle in radians between two line segments which meet at a point is defined as the ratio of the arc length of a circle centered on that point which is cut by the two segments to the radius R of the circle (for example, $\Delta\theta = \Delta s/R$ in Fig. 10-4). Thus, there are $2\pi R/R = 2\pi$ radians in a complete circle, and 2π radians = 360°.] The angle θ is called the **angular displacement** of the rigid body. By convention, the angle θ is considered positive when it is measured counterclockwise from the x axis.

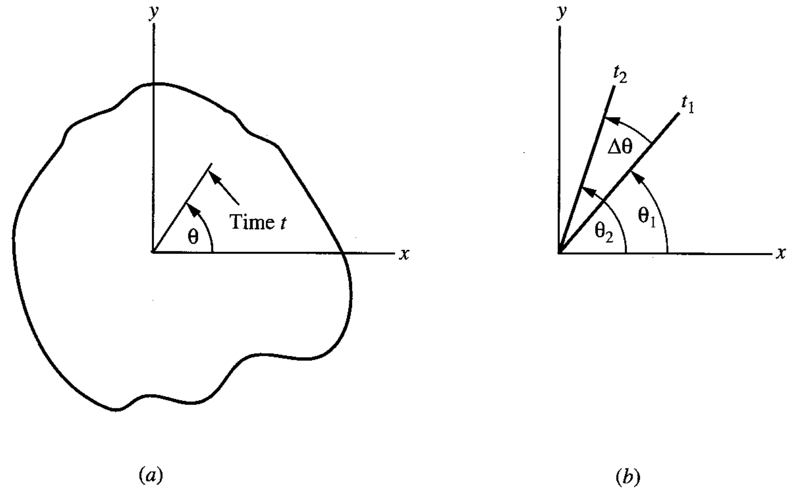

Fig. 10-3

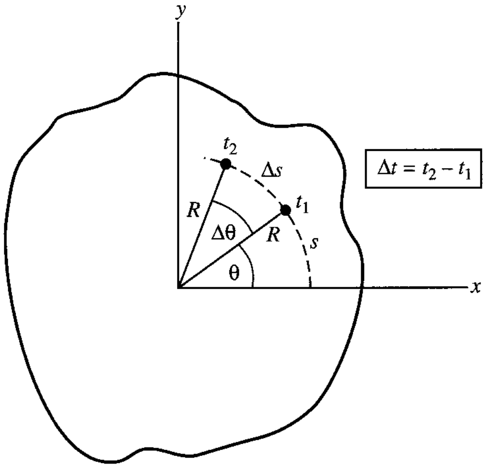

Fig. 10-4

Angular Velocity

To get an idea of how fast the body is rotating, we define the **average angular velocity** in a given time interval, say from t_1 to t_2, as

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} \equiv \frac{\Delta\theta}{\Delta t} \tag{10.1}$$

(See Fig. 10-3(*b*).) Note how this definition mimics that of average velocity in one-dimensional translation, with ω and θ replacing v and x in Eq. (*2.1*). This analogy between straight-line and angular quantities is quite extensive, as will be seen. The units of angular velocity are rad/s.

The **instantaneous angular velocity** at time t_1, is defined as the limit of the average angular velocity as $t_2 \to t_1$ (or, equivalently, as $\Delta t \to 0$):

$$\omega(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \tag{10.2}$$

Note that ω can be positive or negative, depending on whether θ is increasing or decreasing in time. For our angle convention, ω is positive for counterclockwise rotation and negative for clockwise rotation.

Problem 10.1.

(*a*) A body is rotating at a constant angular velocity of 4.0 rad/s. If the angle θ at $t = 0$ was $\theta_0 = 1.50$ rad, find θ at $t = 2.0$ s.

(*b*) What would the answer to (*a*) be if $\omega = -4.0$ rad/s?

Solution

(*a*) If ω is constant we have $\omega = \omega_{av} = (\theta_2 - \theta_1)/(t_2 - t_1)$. Letting $t_1 = 0$ and $\theta_1 = \theta_0$ and dropping the subscript on t_2 and θ_2, we get $\omega = (\theta - \theta_0)/t$ or

$$\theta = \theta_0 + \omega t \tag{i}$$

Substituting the data in (*i*),

$$\theta = 1.50 \text{ rad} + (4.0 \text{ rad/s})(2.0 \text{ s}) = 9.5 \text{ rad} \tag{ii}$$

(*b*) We have

$$\theta = 1.50 \text{ rad} + (-4.0 \text{ rad/s})(2.0 \text{ s}) = -6.5 \text{ rad}$$

which means the guideline is at 6.5 rad measured clockwise from the x axis.

Angular Acceleration

As you probably have guessed, the next thing to do is to define the rate of change of the instantaneous angular velocity, and we again do so in complete analogy to straight-line motion. The **average angular acceleration** α in the time interval from t_1 to t_2 is defined as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \quad (10.3)$$

The **instantaneous angular acceleration** α at time t_1 is then

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \quad (10.4)$$

The units of α are rad/s^2. We note that θ, ω, and α are related to each other in precisely the same way as are x, v, and a. For this reason all formulas of Sec. 2.6 hold for constant angular acceleration if x, v, and a are replaced by θ, ω, and α.

Problem 10.2. Find the general expressions for θ and ω as functions of time for the case of constant angular acceleration α.

Solution

From Chap. 2, for straight-line motion under constant acceleration a, we showed that

$$(i) \quad v = v_0 + at \qquad \text{and} \qquad (ii) \quad x = x_0 + v_0 t + \frac{at^2}{2}$$

where v_0 and x_0 are the initial velocity and displacement at $t = 0$. Thus, for rotation under constant acceleration α, we must have

$$(iii) \quad \omega = \omega_0 + \alpha t \qquad \text{and} \qquad (iv) \quad \theta = \theta_0 + \omega_0 t + \frac{\alpha t^2}{2}$$

where ω_0 and θ_0 are the initial angular velocity and angular displacement at $t = 0$.

Problem 10.3. A wheel spinning on an axis through its center has a constant angular acceleration of 3.5 rad/s^2. Assuming the wheel started from rest, find (*a*) the angular velocity after 8.0 s, (*b*) the total angle turned through in 8.0 s.

Solution

(*a*)

$$\omega = \omega_0 + \alpha t = 0 + (3.5 \text{ rad/s}^2)(8.0 \text{ s}) = 28 \text{ rad/s}$$

(*b*) Assume our guideline is located on the x axis at $t = 0$, so we have $\theta_0 = 0$. Then

$$\theta = \theta_0 + \omega_0 t + \frac{\alpha t^2}{2} = \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}(3.5 \text{ rad/s}^2)(8.0 \text{ s})^2 = 112 \text{ rad}$$

Problem 10.4. Through what angle must the wheel in Problem 10.3 turn for the angular velocity to reach $\omega = 140$ rad/s?

Solution

We first solve for the time:

$$\omega = \omega_0 + \alpha t \qquad \text{or} \qquad 140\ \text{rad/s} = 0 + (3.5\ \text{rad/s}^2)t \qquad \text{or} \qquad t = 40\ \text{s}$$

Next we get the angle, noting that, for our case,

$$\theta = \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}(3.5\ \text{rad/s}^2)(40\ \text{s})^2 = 2800\ \text{rad}$$

Problem 10.5.

(*a*) For the case of constant angular acceleration, find an expression for ω directly in terms of θ.

(*b*) Solve Problem 10.4 without first finding the time.

Solution

(*a*) From Chap. 2, we have, for straight-line motion at constant acceleration

$$v^2 = {v_0}^2 + 2a(x - x_0) \qquad (i)$$

By analogy, for fixed-axis rotation

$$\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0) \qquad (ii)$$

(*b*) Applying Eq. (*ii*) and recalling that $\omega_0 = \theta_0 = 0$, we get $\omega^2 = 2\alpha\theta$. Substituting, we get

$$(140\ \text{rad/s})^2 = 2(3.5\ \text{rad/s})\theta \qquad \text{or} \qquad \theta = 2800\ \text{rad}$$

Problem 10.6. How many revolutions of the wheel does the answer of Problem 10.4 or 10.5 correspond to?

Solution

We recall that there are 2π rad in one complete circle. Therefore, the angle turned through equals 2π times the number of revolutions, or

$$2800\ \text{rad} = (6.28)(\text{no. of rev.}) \qquad \text{or} \qquad \text{no. of rev.} = 445.9\ \text{r}$$

Period and Frequency of Uniform Rotation

When an object spins on its axis it returns to any given position every time it rotates through $360° = 2\pi$ rad. The time to make one complete revolution is called the **period** T of the motion. For constant ω the period stays the same from one revolution to the next. Because $\Delta\theta = \omega\Delta t$, we have $2\pi = \omega T$, or

$$T = \frac{2\pi}{\omega} \qquad (10.5)$$

Next, one can define the **frequency** f as the number of revolutions per second, r/s. Therefore, T and f are reciprocals:

$$f = \frac{1}{T} \qquad (10.6)$$

For example, if $T = \frac{1}{2}$ s, then $f = 2$ r/s. From (*10.5*) and (*10.6*) we get

$$f = \frac{\omega}{2\pi} \qquad (10.7)$$

Note. The concepts of period and frequency are useful not only for rotations but for any kind of motion that repeats itself on a regular basis, such as oscillations. The period always refers to the time for one complete repetition of the motion and the frequency to the number of repetitions per second.

Problem 10.7. Find the period and frequency of the motion described in Problem 10.1(*a*).

Solution

$$\omega = 4.0 \text{ rad/s} \Rightarrow T = \frac{2\pi}{\omega} = \frac{6.28 \text{ rad}}{4.0 \text{ s}} = 1.57 \text{ s}$$

$$f = \frac{1}{T} = 0.637 \text{ r/s}$$

Problem 10.8.

(*a*) Find the value of ω_{av} for the wheel of Problem 10.3 over the time interval from when it starts to when it reaches $\omega = 140$ rad/s.

(*b*) Find the period over the same time interval if the wheel had been rotating with constant angular velocity ω_{av}.

Solution

(*a*) By definition, $\omega_{av} = \Delta\theta/\Delta t$. In Problem 10.4 we determined that $\Delta t = 40$ s and $\Delta\theta = 2800$ rad. Then $\omega_{av} = 70$ rad/s.

(*b*) $T =$ (total time)/(number of revolutions). We already know from Problem 10.4 that the total time = 40 s. From Problem 10.6 we have that the corresponding number of revolutions = 445.9 r. Then

$$T = \frac{40 \text{ s}}{445.9 \text{ r}} = 0.0897 \text{ s}$$

[Or, directly from part (*a*), $T = 2\pi/\omega_{av} = (6.28 \text{ rad})/(70 \text{ rad/s}) = 0.0897$ s.]

10.2 KINEMATICS OF INDIVIDUAL PARTICLES IN A ROTATING RIGID BODY

In Fig. 10-4 we depict a particular particle in a rigid body rotating about the z axis. The particle is constrained to move on a circle of radius R, where R is the perpendicular distance from the z axis to the particle. The velocity of the particle must always point along the tangent to the circle. Let s be the arc length of the circle of motion from the x axis to the position of the particle at time t_1, and let θ be the corresponding angle between the x axis and the radius line to the particle at t_1. At a time $t_2 = t_1 + \Delta t$, and additional arc length Δs and a corresponding angle $\Delta\theta$ will be swept through. As can be seen from the figure,

$$s = R\theta \qquad \text{and} \qquad \Delta s = R\,\Delta\theta \qquad (10.8a, b)$$

If we divide both sides of Eq. (*10.8b*) by Δt we get $\Delta s/\Delta t = R\,\Delta\theta/\Delta t$. In the limit as $\Delta t \to 0$ (holding t_1 fixed), we have $\Delta s/\Delta t \to v$, the magnitude of the velocity of the particle as it moves along its circular trajectory. Also, $\Delta\theta/\Delta t \to \omega(t)$, the angular velocity. Thus, in this limit we get

$$v = R\omega \qquad (10.9)$$

The acceleration of the particle consists of two parts. One part is the centripetal acceleration $\mathbf{a}_r$, which is due to the changing direction of the velocity vector **v**, as the particle moves around the circle, and points radially inward along R to the center of the circle of motion with magnitude

$$a_r = \frac{v^2}{R} = \omega^2 R \tag{10.10}$$

where the last expression in Eq. (*10.10*) follows from Eq. (*10.9*). The other part of the acceleration is the tangential acceleration $\mathbf{a}_t$, which is due to the time variation of v, and which points along the tangent to the circle. By (*10.9*)

$$\Delta v = R\,\Delta\omega \qquad \text{whence} \qquad \frac{\Delta v}{\Delta t} = R\,\frac{\Delta\omega}{\Delta t}$$

Letting $\Delta t \to 0$ in the last equation gives

$$a_t = R\alpha \tag{10.11}$$

From Eqs. (*10.9*) to (*10.11*), we see that.the velocity and acceleration of each particle in the body is completely determined from a knowledge of ω and α as well as of the radial distance R to the particle. The kinematic quantities for a typical particle are depicted in Fig. 10-5.

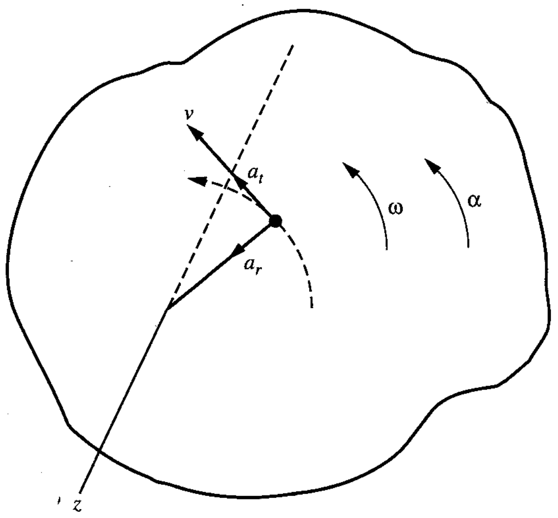

Fig. 10-5

Problem 10.9. A disk of radius 40 cm is spinning at a constant angular velocity of $\omega = 30$ rad/s.

(*a*) Find the velocity of a point on the rim of the disk.

(*b*) Find the acceleration of a point on the rim of the disk.

(*c*) Repeat parts (*a*) and (*b*) for a point one-third of the radius out from the center of the disk.

Solution

(*a*) $v = \omega R = (30\ \text{rad/s})(0.40\ \text{m}) = 12\ \text{m/s}$. (Note that the radian unit does not appear in the final answer because it is dimensionless, and when multiplied by a length it results in a length.)

(*b*) Since there is no angular acceleration, $a_t = 0$, and only the centripetal acceleration a_r contributes:

$$a_r = \omega^2 R = (30\ \text{rad/s})^2\,(0.40\ \text{m}) = 360\ \text{m/s}^2$$

(c) All that changes here is that the radius of the circle for the new particle is $r = R/3$. Therefore both v and a_r are reduced to one-third of the corresponding values for a point on the rim: $v = 4.0$ m/s; $a_r = 120$ m/s^2.

Problem 10.10. Assume that in Problem 10.9 the body has an angular acceleration of 4.5 rad/s^2. Assume also that at the instant in question ω has the same value as in Problem 10.9. How do the answers to Problem 10.9 change?

Solution

Since ω is the same as in Problem 10.9, the velocities and the centripetal accelerations are the same. The only change is that both particles now have a tangential component of acceleration: $a_t = R\alpha = (0.40\text{ m})(4.5\text{ rad/s}^2) = 1.80\text{ m/s}^2$ for the particle on the rim, and $a_t = R\alpha/3 = 0.60\text{ m/s}^2$ for the particle at one-third the radius of the disk.

10.3 DYNAMICS OF A RIGID BODY ROTATING ABOUT A FIXED AXIS

Torque Revisited

As you might suspect from Chap. 9, the torque (or moment) of a force that acts on a rigid body is going to play a major role in the rotation of that body. Before showing this, however, we need to take a more sophisticated look at torque. In Chap. 9 we dealt only with coplanar forces and defined the torque as a positive or negative algebraic quantity. Actually, torque is generally defined as a three-dimensional vector quantity $\mathbf{\Gamma}$. Figure 10-6 shows a force $\mathbf{F}$ acting on a particle located at displacement $\mathbf{r}$ from the origin, where $\mathbf{F}$ is not necessarily in the xy plane. If θ is the angle between $\mathbf{F}$ and $\mathbf{r}$ (tail to tail), then the magnitude of the torque (on the particle) *about the origin* is defined as

$$|\mathbf{\Gamma}| = rF\sin\theta \qquad (10.12)$$

The vector $\mathbf{\Gamma}$ is, by definition, directed perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{F}$, as shown in Fig. 10-6. Note that there are two antiparallel choices for the direction perpendicular to a plane, and we must remove that ambiguity in the definition of $\mathbf{\Gamma}$. We agree that the force $\mathbf{F}$ will tend to rotate an object (pinned at the origin) counterclockwise about the positive $\mathbf{\Gamma}$ direction. An equivalent rule is to let the curved fingers of your right hand swing $\mathbf{r}$ through angle θ into $\mathbf{F}$; then your extended thumb points in the direction of $\mathbf{\Gamma}$. If $\mathbf{r}$ and $\mathbf{F}$ were in the xy plane, $\mathbf{\Gamma}$ would clearly point in the $\pm z$ direction. Indeed from Eq. (*10.12*) we see that the z component of such a torque would be $\Gamma_z = \pm rF\sin\theta$. By our convention, the sign is determined by whether the force tends to give rise to a counterclockwise or clockwise rotation about the positive z axis. Since all the forces in Chap. 9 were in the xy plane, the definition of torque given there was really the z component of the vector torque, as defined here.

When the forces acting on a rigid body are not coplanar, the vector torques can point in different directions, and the resultant torque can be quite complex. We will not deal with such general situations. We will restrict ourself to the effect of forces on a rigid body that is free to rotate about a fixed axis. For definiteness we will assume the fixed axis is the z axis of our coordinate system. The following statements can be shown to hold:

1. Only the z component of the torque due to a force has any effect on the rotation of a rigid body about the z axis. (The z component of the torque is often called the *torque about the z axis*.)

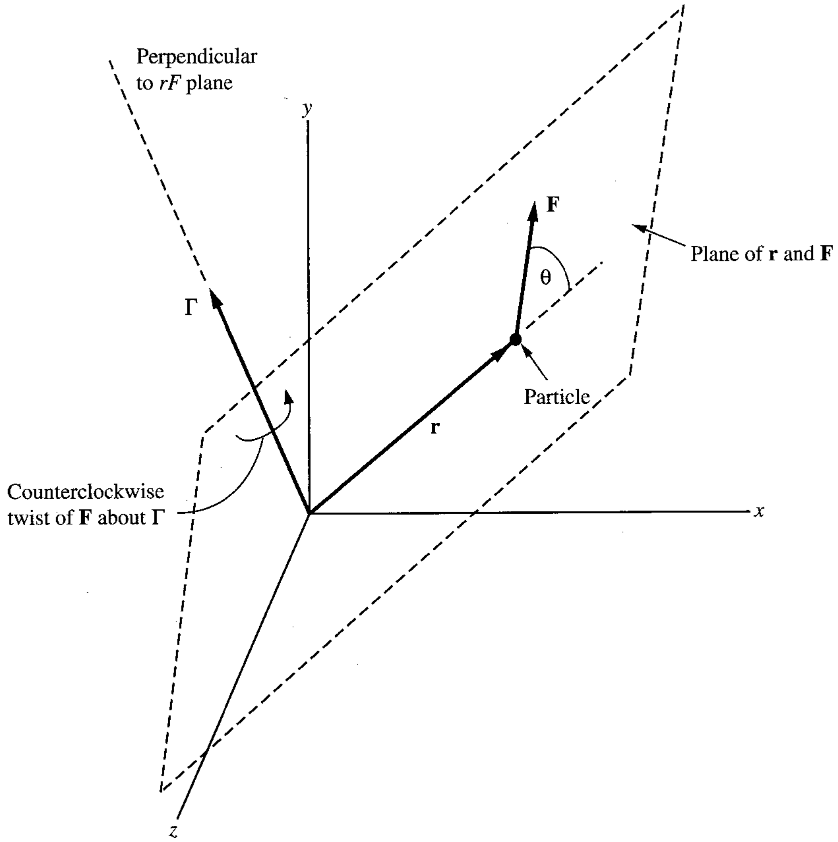

Fig. 10-6

2. Only the x and y components of a force $\mathbf{F}$ can contribute to the z component of a torque. Thus any force (or part of a force) that points along the z axis cannot contribute to torque about that axis.
3. The z component of the torque, due to a force $\mathbf{F}$ in the xy plane, is given by $\Gamma_z = rF \sin\theta$, where r stands for the *perpendicular distance* from the z axis to the point of application of the force and θ is the angle $\mathbf{F}$ and a line along r.

We will drop the subscript z in what follows, so Γ has the same meaning as in Chap. 9.

Torque, Angular Acceleration, and the Dynamic Laws of Rotation

In Fig. 10-7 we examine the ith particle, of mass m_i, in a rigid body that is rotating about the z axis. We assume the resultant force acting on that particle is $\mathbf{F}_i$, which includes both the forces due to all the other particles in the body and the external forces from outside the body. From Newton's second law, $\mathbf{F}_i = m_i\mathbf{a}_i$, where $\mathbf{a}_i$ is the acceleration of m_i. Thus, in the tangential direction, we must have

$$F_{it} = m_i a_{it} = m_i r_i \alpha \tag{10.13}$$

where the last step follows from *(10.11)*. Recognizing that $\Gamma_i = r_i F_{it}$ is the z component of the torque of the force $\mathbf{F}_i$, we find from *(10.13)* that $\Gamma_i = m_i r_i^2 \alpha$. Repeating the same procedure for every particle of the body and adding up all the Γ_i, we get

$$\Gamma_T = \sum \Gamma_i = \sum m_i r_i^2 \alpha = \left(\sum m_i r_i^2\right)\alpha \tag{10.14}$$

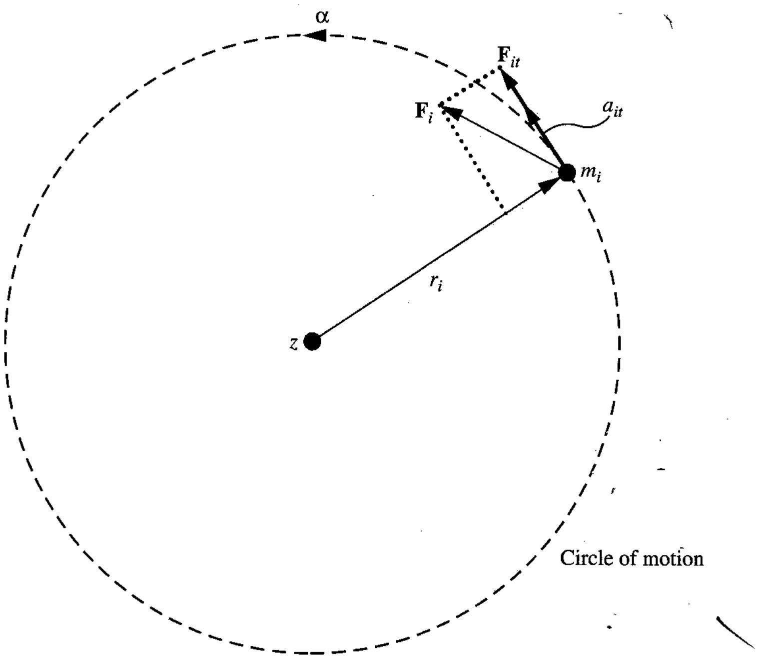

Fig. 10-7

where the sum goes over all N particles of the body, i.e., from $i = 1$ to N. The last term on the right follows since α is the same for all particles in the body and can be factored out of the sum. We define the **moment of inertia** I of the body about the z axis as

$$I = \sum m_i r_i^2 \qquad (10.15)$$

From definition (*10.15*), we see that I has dimensions of mass times distance squared = $\mathrm{kg \cdot m^2}$ or $\mathrm{g \cdot cm^2}$ or $\mathrm{slug \cdot ft^2}$. The left side of Eq. (*10.14*) includes all the torques on all the particles, including the torques that the particles exert on each other. It is easy to see from Newton's third law that all these *internal* torques must add up to zero. This is shown in Fig. 10-8, where we examine the torques due to the action-reaction pair of forces on two arbitrary particles m_i and m_j. Since $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$ are equal and opposite and have the same line of action, they must give rise to equal and opposite torques. Since all the internal torques are due to action-reaction pairs, all the internal torques must cancel in pairs. Thus, the only torque left in Γ_T is Γ_{ext}, the resultant torque due to external forces. Using Eqs. (*10.14*) and (*10.15*), we get

$$\Gamma_{\text{ext}} = I\alpha \qquad \text{or} \qquad \Gamma = I\alpha \qquad (10.16)$$

where on the right we have dropped the subscript, it being understood that we are referring to the resultant external torque. Note the analogy to the dynamic equation for one-dimensional motion $\mathbf{F} = m\mathbf{a}$. Equation (*10.16*) is Newton's second law for rotation about a fixed axis. Note that the quantity $I\alpha$ has dimensions $\mathrm{kg \cdot m^2/s^2} = (\mathrm{kg \cdot m/s^2})\mathrm{m} = \mathrm{N \cdot m}$. This is just the dimensions of torque, as required by Eq. (*10.16*). Table 10.1 is a table of analogs developed thus far between quantities related to rotational motion and translational (one-dimensional) motion.

Problem 10.11. Assuming that the analogies in the table can be extended further, find the rotational analog of the following equations: (*a*) $E_k = \frac{1}{2}mv^2$; (*b*) work = $F\Delta x = \Delta E_k$; (*c*) $P = Fv$; (*d*) $\mathrm{F}\Delta t = \Delta(mv)$.

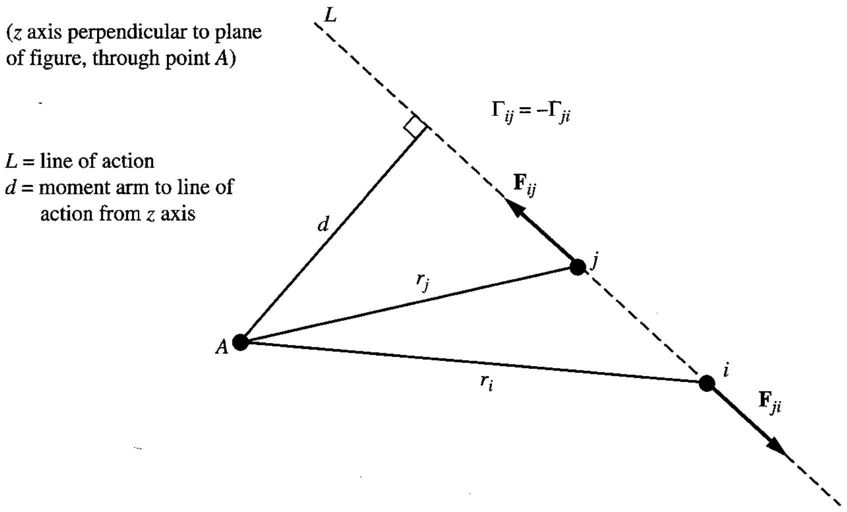

Fig. 10-8

Table 10.1

Rotation (Fixed-Axis)	Translation (One-Dimensional)
θ	x
ω	v
α	a
Γ	F
I	m
$\Gamma = I\alpha$	$F = ma$

Solution

(*a*) $E_k = \frac{1}{2}I\omega^2$; (*b*) work $= \Gamma\Delta\theta = \Delta E_k$; (*c*) $P = \Gamma\omega$; (*d*) $\Gamma\Delta t = \Delta(I\omega)$.

Note. The solutions to Problem 10.11 can all be shown to be true. $\Gamma\Delta\theta$ is the work done by the external forces in rotating a rigid body through an infinitesimal angle $\Delta\theta$, and the relation $\Gamma\Delta\theta = \Delta E_k$ is just the work-energy theorem applied to a rotating object. The power P is then just $\Gamma\Delta\theta/\Delta t = \Gamma\omega$, as expected. The quantity $\Gamma\Delta t$ is called the **angular impulse**, and the quantity $I\omega$ is called the **angular momentum**. The relationship $\Gamma\Delta t = \Delta(I\omega)$ is just the rotational analog of the impulse momentum theorem for linear motion.

Problem 10.12. Show that $E_k = \frac{1}{2}I\omega^2$, the solution of Problem 10.11(*a*), is the kinetic energy of a rigid body rotating about a fixed axis.

Solution

The rotational kinetic energy is given by $E_k = \sum(\frac{1}{2}m_iv_i^2)$, where the sum extends over all the particles of the rotating body, $i = 1 \rightarrow N$. We note that for the *i*th particle, $v_i = r_i\omega$. Substituting into our equation we get $E_k = \sum(\frac{1}{2}m_i r_i^2 \omega^2)$. Since ω is the same for all the terms in the sum, we can factor out ω^2 (and also the factor $\frac{1}{2}$), so we get $E_k = \frac{1}{2}(\sum m_i r_i^2)\omega^2 = \frac{1}{2}I\omega^2$, as required.

The other three results of Problem 10.11 may be verified similarly.

Calculating Moments of Inertia

To solve problems in rotational dynamics it is often necessary to determine the moments of inertia of the various rigid bodies about their axes of rotation.

Problem 10.13. A rigid body consists of four masses held at the corners of a rectangle by rigid bars of negligible mass (Fig. 10-9). Find the moment of inertia about (*a*) an axis through side 1; (*b*) an axis through side 2; (*c*) an axis parallel to side 1 through the geometric center of the object A.

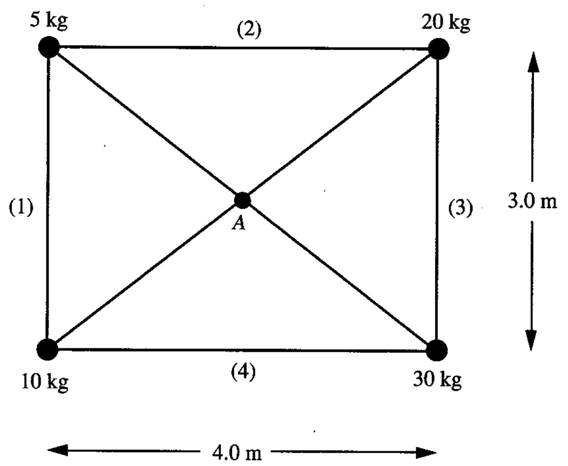

Fig. 10-9

Solution

(*a*) The 5- and 10-kg masses lie on the axis, so they don't contribute to the moment of inertia. The perpendicular distances to the 20- and 30-kg masses are each 4.0 m. Thus we get

$$I_1 = (20\text{ kg})(4.0\text{ m})^2 + (30\text{ kg})(4.0\text{ m})^2 = 800\text{ kg}\cdot\text{m}^2$$

(*b*) Here the 5- and 20-kg masses don't contribute, and so

$$I_2 = (10\text{ kg})(3.0\text{ m})^2 + (30\text{ kg})(3.0\text{ m})^2 = 360\text{ kg}\cdot\text{m}^2$$

(*c*) Each mass is 2.0 m from the axis.

$$I_{1A} = (5\text{ kg})(2.0\text{ m})^2 + (10\text{ kg})(2.0\text{ m})^2 + (20\text{ kg})(2.0\text{ m})^2 + (30\text{ kg})(2.0\text{ m})^2 = 260\text{ kg}\cdot\text{m}^2$$

Problem 10.14. For the situation in Fig. 10-9, find the moment of inertia I_A about an axis through point A, and perpendicular to the plane of the figure.

Solution

The common distance of the four masses from the axis is one-half the diagonal. Since either diagonal and two connecting sides form a right triangle, we can use the Pythagorean theorem to obtain the diagonal length. In our case the sides are of length 3.0 and 4.0 m, respectively, so the diagonal has length 5.0 m. The distance to each mass is thus 2.5 m. Then

$$I_A = (5.0\text{ kg} + 10\text{ kg} + 30\text{ kg} + 20\text{ kg})(2.5\text{ m})^2 = 406\text{ kg}\cdot\text{m}^2$$

Problem 10.15. Figure 10-10 depicts a double wheel whose mass is concentrated in two concentric rims of equal mass, $M_1 = M_2 = 8.0$ kg. The rims are supported by a set of light spokes joining at the geometric center of the wheels. The radii of the two rims are $R_1 = 2.0$ m and $R_2 = 4.0$ m. Find (*a*) the moment of inertia of each rim about a perpendicular axis through the center of the wheel, (*b*) the total moment of inertia of the double wheel about the axis.

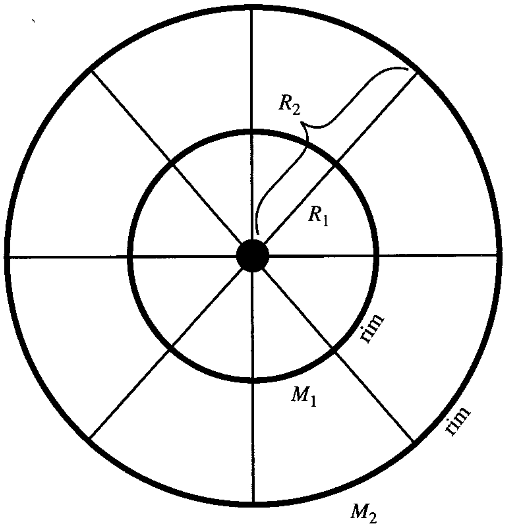

Fig. 10-10

Solution

(*a*) For the first rim, all the particles are at the same distance from the axis, so summing over all particles in this rim,

$$I_1 = (\sum m_i r_i^2) = (\sum m_i)R_1^2 = M_1 R_1^2 = (8.0\text{ kg})(2.0\text{ m})^2 = 32\text{ kg}\cdot\text{m}^2$$

Similarly for the outer rim, we sum over all particles in that rim:

$$I_2 = (\sum m_i r_i^2) = (\sum m_i)R_2^2 = M_2 R_2^2 = (8.0\text{ kg})(4.0\text{ m})^2 = 128\text{ kg}\cdot\text{m}^2$$

(*b*) The total moment of inertia is just the sum of the contributions from the two rims:

$$I_T = 32\text{ kg}\cdot\text{m}^2 + 128\text{ kg}\cdot\text{m}^2 = 160\text{ kg}\cdot\text{m}^2$$

In general, obtaining the moment of inertia of even a simple, symmetric object is possible only by using the calculus. In Fig. 10-11 we show the moments of inertia for a few standard symmetric objects of uniform density: (*a*) a uniform disk of mass M and radius R about a perpendicular axis through its center; (*b*) a uniform cylindrical annulus (a hollowed-out concentric cylinder) of inner radius R_1 and

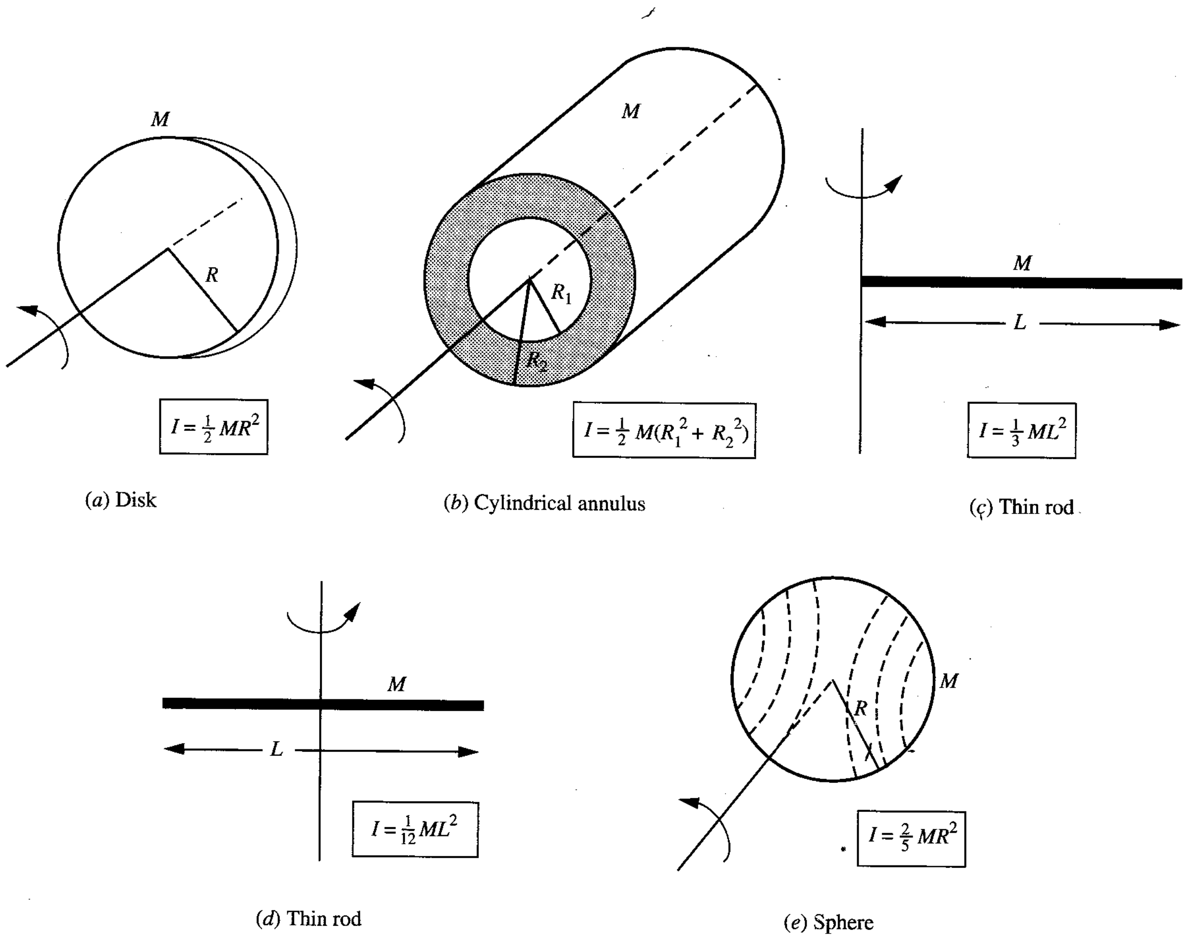

Fig. 10-11 Moments of inertia of uniform bodies.

outer radius R_2 and mass M about the symmetry axis; (c) a thin uniform rod of mass M and length L about a perpendicular axis through one end; (d) a thin uniform rod of mass M and length L about a perpendicular axis through its center; (e) a uniform sphere of mass M and radius R about an axis through its center.

Solving Dynamics Problems

We are now prepared to solve a variety of problems involving rigid-body rotation.

Problem 10.16. A uniform disk of mass $M = 12.0$ kg and radius $R = 0.75$ m (Figure 10-12) is free to rotate about a frictionless axle through its center. A cord is wrapped around the rim of the disk and is pulled with a steady tension of $T = 100$ N. As the cord is unwound, the disk rotates, and we assume there is no slippage between the cord and the rim of the disk. If the disk starts from rest, find the angular velocity and the angle turned through after 6.0 s.

Solution

The only external forces acting on the disk are the tension T in the cord, the weight and the normal force due to the axle. The weight and normal force have no moment about the axis of rotation, so they do not contribute to the torque. The tension acts tangential to the rim, so the moment arm is just the radius

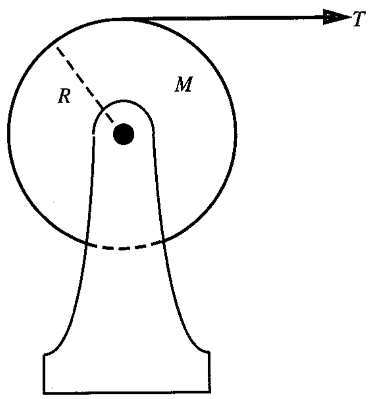

Fig. 10-12

of the disk. Then, from Eq. (*10.16*) and Fig. 10-11 we have $\Gamma = I\alpha$ or $TR = (\frac{1}{2}MR^2)\alpha$. Substituting, we get

$$(100\ \text{N})(0.75\ \text{m}) = (6.0\ \text{kg})(0.75\ \text{m})^2\alpha \qquad \text{or} \qquad \alpha = 22.2\ \text{rad/s}^2$$

Since α is constant, and the system starts from rest, we have

$$\omega = \alpha t = (22.2\ \text{rad/s}^2)(6.0\ \text{s}) = 133\ \text{rad/s}$$

Similarly, assuming we start measuring θ from $t = 0$, we have

$$\theta = \tfrac{1}{2}\alpha t^2 = (11.1\ \text{rad/s}^2)(6.0\ \text{s})^2 = 400\ \text{rad}$$

Problem 10.17.

(*a*) Calculate the work W done by the tension T of Problem 10.16, in rotating the disk through 400 rad.

(*b*) Using the work-energy theorem, find the angular velocity ω of the disk after it rotates through 400 rad., and compare to the result of Problem 10.16.

Solution

(*a*) Since the torque is constant at 75 N · m, we have [Problem 10.11(*b*)]

$$\text{Work} = W = \Gamma\theta = (75\ \text{N}\cdot\text{m})(400\ \text{rad}) = 30{,}000\ \text{J}$$

[Or, we could calculate the work as $W = Tx$, where x is the distance through which the cord is pulled as the disk rotates through 400 rad. Now x is just the length of cord that unwinds from the rim, so $x = R\theta = (0.75\ \text{m})(400\ \text{rad}) = 300\ \text{m}$. Then $W = (100\ \text{N})(300\ \text{m}) = 30{,}000\ \text{J}$ as before.]

(*b*) Since the disk starts from rest, we have $W = \frac{1}{2}I\omega^2$. We recall that $I = \frac{1}{2}MR^2 = (6.0\ \text{kg})(0.75\ \text{m})^2 = 3.375\ \text{kg}\cdot\text{m}^2$. Then

$$30{,}000\ \text{J} = \tfrac{1}{2}(3.375\ \text{kg}\cdot\text{m}^2)\omega^2 \qquad \text{or} \qquad \omega = 133\ \text{rad/s}$$

which is the same result as in Problem 10.16.

Problem 10.18. A block of mass $m = 3.0$ kg hangs from one end of a cord, while the other end is wrapped around a uniform wheel of radius $R = 30$ cm and moment of inertia $I = 0.80\ \text{kg}\cdot\text{m}^2$. The wheel is free to rotate about a frictionless horizontal axle through its center, as shown in Fig. 10-13(*a*). The system starts from rest and the cord unravels with no slippage.

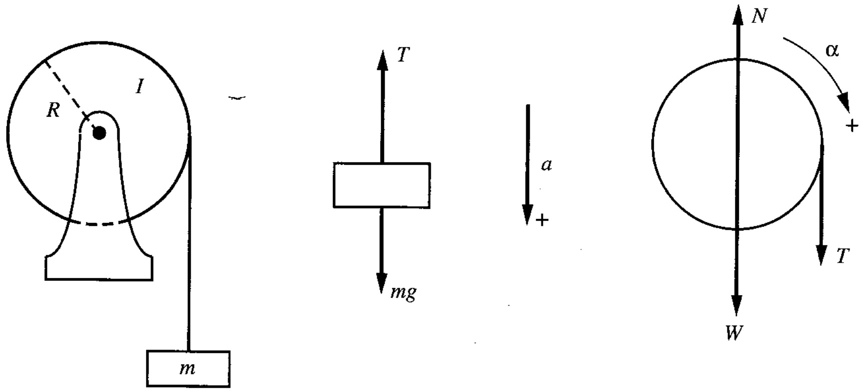

Fig. 10-13

(*a*) Find the acceleration a of the block.

(*b*) Find the angular velocity of the wheel after the block has dropped through a distance of 1.50 m.

Solution

(*a*) We first apply Newton's second law to the block. The free-body diagram, Fig. 10-13(*b*), gives

$$mg - T = ma \qquad (i)$$

This equation has two unknowns, the tension T and the acceleration a. We therefore need another equation, and we turn to the rotational law for the wheel. The body diagram for the wheel is shown in Fig. 10-13(*c*). For this problem we find it convenient to choose *clockwise* as our positive rotational direction—which is okay as long as we do so consistently for all rotational quantities, including the torque. We note that neither the normal force at the axle nor the weight of the wheel W, which acts at the center, can contribute to the torque about the axis of rotation. The only torque is due to the tension in the cord, and is clockwise, giving $\Gamma = TR$. Then

$$\Gamma = I\alpha \Rightarrow TR = I\alpha \qquad (ii)$$

Equation (*ii*) has two unknowns, T and α. Thus between (*i*) and (*ii*) we have two equations with three unknowns. We have, in addition, a relationship between a and α. Since there is no slippage between the rim and the cord, the velocity of the cord v (which is the same as the velocity of the block), must always equal the velocity of a point on the rim of the wheel. Similarly, the acceleration of the cord a (which is the same as the acceleration of the block) must equal the tangential acceleration of a point on the rim. Thus we have

$$v = R\omega \qquad a = R\alpha \qquad (iiia, b)$$

Equation (*ii*) can then be rewritten as $TR = Ia/R$ or

$$T = \left(\frac{I}{R^2}\right) a \qquad (iv)$$

Note that this looks just like Newton's second law for an object of mass $M' = I/R^2$ moving along a horizontal frictionless surface under the action of a horizontal force T. We can now solve Eqs. (*i*) and (*iv*) to obtain T and a. Adding, we get

$$mg = \left(\frac{I}{R^2} + m\right) a \qquad (v)$$

Substituting numerical values into (v) we get

$$(3.0\ \text{kg})(9.8\ \text{m/s}^2) = \left[\frac{0.80\ \text{kg}\cdot\text{m}}{(0.30\ \text{m})^2} + 3.0\ \text{kg}\right]a \qquad \text{or} \qquad a = 2.47\ \text{m/s}^2$$

(*b*) The system starts from rest, and we can set $y = 0$ at the initial position of the block. Then,

$$v^2 = 2ay = 2(2.47\ \text{m/s}^2)(1.5\ \text{m}) = 7.41\ \text{m}^2/\text{s}^2 \qquad \text{or} \qquad v = 2.72\ \text{m/s}$$

Using (*iiia*), we have

$$\omega = \frac{v}{R} = \frac{2.72\ \text{m/s}}{0.30\ \text{m}} = 9.07\ \text{rad/s}$$

Problem 10.19. Check Problem 10.18(*b*), using energy conservation.

Solution

The only external force that does work is gravity. (Note that the tension in the cord does positive work on the wheel but does an equal and opposite amount of work on the block, thus doing no net work on the system as a whole. This is to be expected, since the cord is not a source of energy but merely a device to transfer energy from one part of the system to the other.) The conservation of energy for the system as a whole then requires that the loss in gravitational potential energy as the block falls equals the gain in kinetic energy of the system. For our case, starting from rest, $mgy = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Recalling that $v = \omega R$, we have $mgy = \frac{1}{2}(mR^2 + I)\omega^2$. Substituting, we set

$$(3.0\ \text{kg})(9.8\ \text{m/s}^2)(1.5\ \text{m}) = \tfrac{1}{2}[(3.0\ \text{kg})(0.30\ \text{m})^2 + 0.80\ \text{kg}\cdot\text{m}^2]\omega^2$$

Solving, we get $\omega = 9.08$ rad/s, which matches the result in Problem 10.18.

Problem 10.20. The engine of an automobile delivers energy to the wheel via the driveshaft. Assuming that an engine delivers 180 hp while the driveshaft is spinning at a frequency of $f = 80$ r/s, find the torque exerted by the engine on the driveshaft.

Solution

We have $P = \Gamma\omega = \Gamma(2\pi f)$, so

$$(180\ \text{hp})(550\ \text{ft}\cdot\text{lb/hp}) = \Gamma(6.28\ \text{rad/r})(80\ \text{r/s}) \qquad \text{or} \qquad \Gamma = 197\ \text{lb}\cdot\text{ft}$$

Problem 10.21. For Problem 10.18, find (*a*) the tension in the cord, (*b*) the instantaneous power delivered via the cord at the moment the block has fallen 1.50 m.

Solution

(*a*) From Problem 10.18, Eq. (*i*), we have $mg - T = ma$ or

$$T = m(g - a) = (3.0\ \text{kg})(9.8\ \text{m/s}^2 - 2.47\ \text{m/s}^2) = 22.0\ \text{N}$$

(*b*) $P = \Gamma\omega = TR\omega$. Substituting, we get [using the results of Problem 10.18(*b*)]

$$(22.0\ \text{N})(0.30\ \text{m})(9.07\ \text{rad/s}) = 59.9\ \text{W}$$

10.4 ANGULAR MOMENTUM

General Definition

We have already defined the angular momentum of a rigid body rotating about a fixed axis to be $I\omega$. The more general definition, however, like that of torque, is a vector quantity. Moreover, the formal definition is made in terms of a particle moving through space, not a rotating rigid body, and at first glance this definition appears to have nothing to do with $I\omega$. The definition of vector angular momentum is completely analogous to that of vector torque. The situation is shown in Fig. 10-14. There a particle of mass m moves with velocity $\mathbf{v}$ (not necessarily in the xy plane) and thus has momentum $\mathbf{p} = m\mathbf{v}$. At the instant in question the displacement of the particle from the origin is $\mathbf{r}$. The magnitude of the angular momentum $\mathbf{l}$ *about the origin* is defined as

$$l = rmv \sin\theta = rp \sin\theta \tag{10.17}$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{p}$.

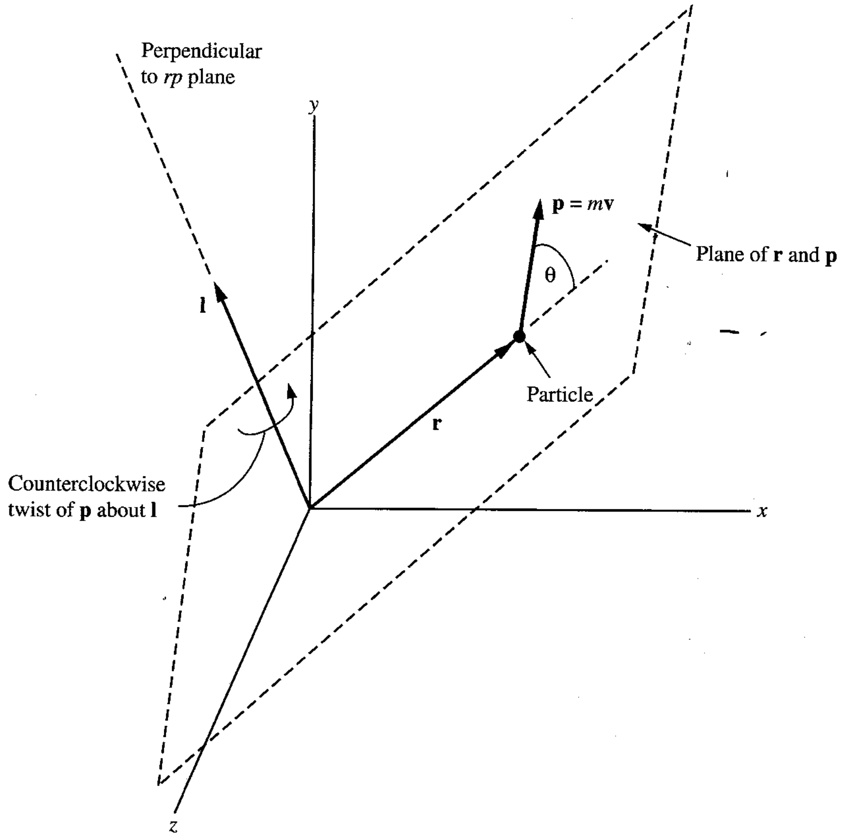

Fig. 10-14

The direction of $\mathbf{l}$ is defined to be perpendicular to the plane of $\mathbf{r}$ and $\mathbf{p}$, and pointing in the same direction as a torque would point if $\mathbf{p}$ were a force instead of the momentum. Thus, for a particle moving in the xy plane, the angular momentum would point in the $\pm z$ direction, with the sign determined by whether the particle appears to be going "around" the z axis counterclockwise or clockwise. (Again, the standard convention is that counterclockwise is positive.) Because of the complete analogy of the definition of angular momentum to that of the moment of a force, the angular momentum is often called the **moment of momentum**.

Problem 10.22.

(*a*) A particle of mass $m = 2.5$ kg moves in the xy plane with a velocity $\mathbf{v}$ of magnitude $v = 30.0$ m/s parallel to the x axis, as shown in Fig. 10-15. If the perpendicular distance from the origin to the

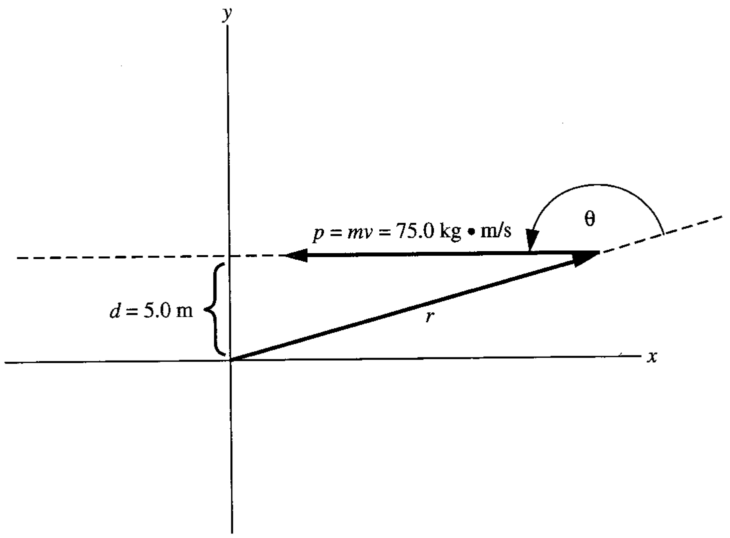

Fig. 10-15

line of motion is $d = 5.0$ m, as shown, find the magnitude and direction of the angular momentum **l** about the origin. Assume z is positive out of the paper.

(*b*) If the particle continues to move with uniform velocity, how will the angular momentum change with time?

Solution

(*a*) The direction of **l** is along the positive z axis, because the momentum vector, if thought of as a force, would give rise to a counterclockwise moment about the positive z axis. The magnitude of the angular momentum is given by $l = rmv \sin\theta$. We note that $r \sin\theta = r \sin(180° - \theta) = d$, so

$$l = dmv = (5.0\ \text{m})(2.5\ \text{kg})(30.0\ \text{m/s}) = 375\ \text{kg}\cdot\text{m}^2/\text{s}$$

(*b*) For any position along the straight-line path of motion we have $r \sin\theta = d$. Also, mv remains constant by hypothesis. Thus $l = rmv \sin\theta = dmv = \text{constant}$. The direction of **l** also remains constant, pointing along the positive z axis.

Rotation of a Rigid Body about a Fixed Axis

For a rigid body rotating about the z axis, all the particles are moving in concentric circles in planes parallel to the xy plane. The z component of the angular momentum of the ith particle, in analogy to the z component of a torque, can be shown to depend only on the perpendicular displacement r_i from the z axis to the particle and on the momentum of the particle in the xy plane. It is given by $l_{iz} = r_i m_i v_i \sin\theta_i$, where θ_i is the angle between the line r_i and the vector momentum $\mathbf{v}_i$. (For definiteness, we assume the object is rotating counterclockwise about the z axis, so l_{iz} has a plus sign.)

Since the ith particle is moving on a circle of radius r_i, the angle $\theta_i = 90°$ so that $\sin\theta_i = 1$ and $l_{iz} = m_i v_i r_i$. Since $v_i = r_i\omega$, this becomes $l_{iz} = m_i r_i^2\omega$. The total z component of angular momentum is $L_z = \sum l_{iz} = \sum m_i r_i^2\omega$. Since ω is the same for all particles in the body, and $I = \sum m_i r_i^2$, we get

$$L_z = I\omega \qquad (10.18a)$$

Thus, our earlier definition of angular momentum, for a rigid body rotating about a fixed axis, is really the component of the vector angular momentum along that axis. When discussing rotation about the z axis some texts drop the z subscript and write Eq. (*10.18a*) as

$$L = I\omega \tag{10.18b}$$

> ***Note.*** For a rigid body that is symmetrical about the axis of rotation, it can be shown that the *only* component of angular momentum is the component along the axis.

Problem 10.23.

(*a*) A uniform sphere of radius $R = 30$ cm and mass $M = 15$ kg spins about the z axis with a counterclockwise angular velocity of $\omega = 20$ rad/s. Find the z component of angular momentum of the sphere.

(*b*) An external force applied to the sphere exerts a counterclockwise torque about the z axis of $\Gamma = 10\ \text{N}\cdot\text{m}$ for a period of 3.0 s. The force then changes so that it exerts a clockwise torque of $\Gamma = 20\ \text{N}\cdot\text{m}$ for the next 4.0 s. Find the total angular impulse in the 7.0-s interval.

(*c*) Find the angular velocity of the sphere at the end of the 7.0-s interval.

Solution

(*a*) $L = I\omega$. $I = \frac{2}{5}MR^2 = \frac{2}{5}(15\ \text{kg})(0.30\ \text{m})^2 = 0.54\ \text{kg}\cdot\text{m}^2$. Then $L = (0.54\ \text{kg}\cdot\text{m}^2)(20\ \text{rad/s}) = 10.8\ \text{J}\cdot\text{s}$.

(*b*) Angular impulse $= \Gamma_1\,\Delta t_1 + \Gamma_2\,\Delta t_2 = (10\ \text{N}\cdot\text{m})(3.0\ \text{s}) + (-20\ \text{N}\cdot\text{m})(4.0\ \text{s}) = -50\ \text{J}\cdot\text{s}$.

(*c*) Angular impulse $= \Delta(I\omega) = I\,\Delta\omega = I(\omega_f - \omega_i)$, since I is constant. Then

$$-50\ \text{J}\cdot\text{s} = (0.54\ \text{kg}\cdot\text{m}^2)(\omega_f - 20\ \text{rad/s}) \qquad \text{or} \qquad \omega_f = -72.6\ \text{rad/s}$$

Conservation of Angular Momentum

The general law of conservation of angular momentum for an arbitrarily moving system of particles is a rather complicated affair, but it is completely analogous to conservation of linear momentum for such a system. It may be stated as follows: If the resultant external vector torque (about the origin) $\mathbf{\Gamma}$ for a system of particles is zero, then the vector sum of the angular momenta of all the particles, $\mathbf{L} = \sum \mathbf{l}_i$, stays constant in time.

For the special case of objects rotating about a fixed axis, say the z axis, the law takes on a simpler form: If the total external torque about the axis is zero, then the total z component of angular momentum doesn't change.

Problem 10.24. Two disks are freely rotating about a common frictionless axle, as shown in Fig. 10-16. The disks rest on frictionless pins in the axle. If the pin under the upper disk is suddenly removed, without otherwise disturbing the disk, the disk will fall onto the lower disk, and after a short time the friction between the two disks will bring them to a common angular velocity.

(*a*) Prove from basic principles that angular momentum is conserved in this case.

(*b*) Calculate the final angular velocity of the combination for the data of Fig. 10-16.

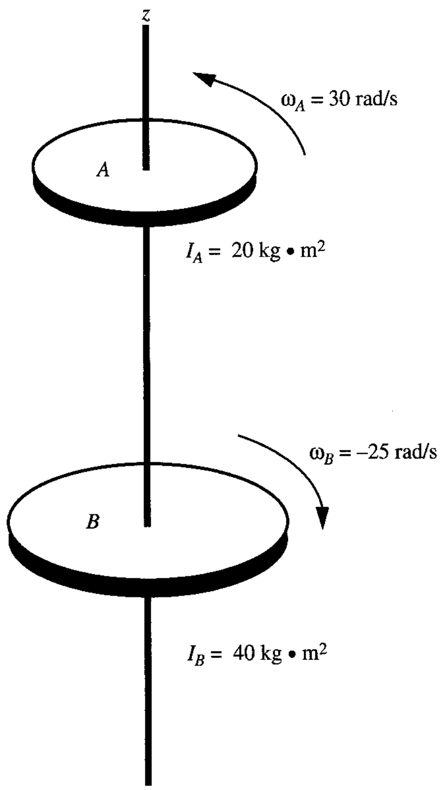

Fig. 10-16

Solution

(*a*) The external forces—the force of gravity and the normal force of the lower pin—do not contribute to the torque about the *z* axis. The only torques on the two disks as they collide are the frictional torques they exert on each other. These torques are equal and opposite, as a consequence of Newton's third law: $\boldsymbol{\Gamma}_{AB}$ (torque of disk *A* on disk *B*) = $-\boldsymbol{\Gamma}_{BA}$. For any infinitesimal time interval Δt, we have $\boldsymbol{\Gamma}_{AB}\,\Delta t = \Delta(I_B\omega_B)$ and $\boldsymbol{\Gamma}_{BA}\,\Delta t = \Delta(I_A\omega_A)$. Thus

$$\boldsymbol{\Gamma}_{AB} = -\boldsymbol{\Gamma}_{BA} \Rightarrow \Delta(I_A\omega_A) = -\Delta(I_B\omega_B)$$

Since the changes in angular momentum of the two disks are equal and opposite, the combined angular momentum of the two disks must remain constant:

$$(I_A + I_B)\omega_f = I_A\omega_{Ai} + I_B\omega_{Bi} \qquad (i)$$

(*b*) Setting the total angular momentum after collision equal to the total before, we obtain

$$[(20\ \text{kg}\cdot\text{m}^2) + (40\ \text{kg}\cdot\text{m}^2)]\omega_f = (20\ \text{kg}\cdot\text{m}^2)(30\ \text{rad/s}) + (40\ \text{kg}\cdot\text{m}^2)(-25\ \text{m/s})$$

from which $\omega_f = -6.67$ rad/s.

Problem 10.25. A student stands on a light platform that is free to rotate without friction about a vertical axis. His arms are initially outstretched, as shown in Fig. 10-17(*a*), and he holds a 5.0-kg mass in each hand. His initial angular velocity is $\omega_i = 4.0$ rad/s.

(*a*) What is the new angular velocity ω_f when his elbows are bent as shown in Fig. 10-17(*b*)? [*Data*: The moment of inertia of the student (without the weights) about the rotation axis, with his arms outstretched, is 5.0 $\text{kg}\cdot\text{m}^2$; with his elbows bent, it is 4.5 $\text{kg}\cdot\text{m}^2$. The weights are initially 0.90 m from the axis of rotation, and finally 0.30 m.]

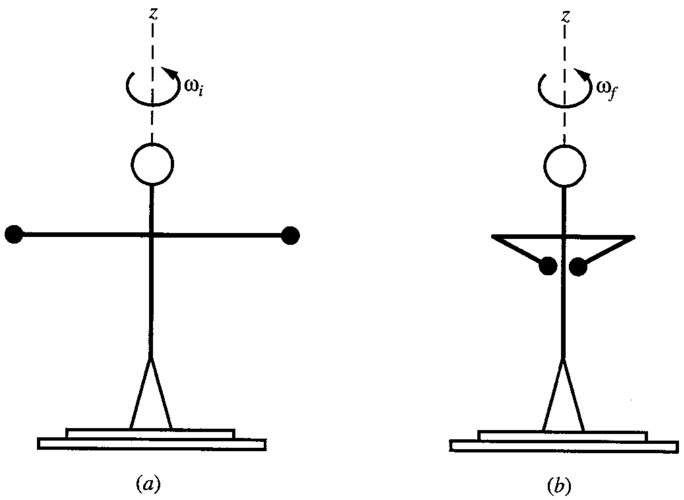

Fig. 10-17

(*b*) What is the increase in kinetic energy of the rotating system?

(*c*) Where does the extra energy in part (*b*) come from?

Solution

(*a*) Since no external torque about the axis of rotation acts on the system consisting of the student, the masses, and the light platform, angular momentum of the system about that axis is conserved: $I_i\omega_i = I_f\omega_f$. From the data,

$$I_i = 5.0\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(0.90\ \text{m})^2 = 13.1\ \text{kg}\cdot\text{m}^2$$
$$I_f = 4.5\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(0.30\ \text{m})^2 = 5.4\ \text{kg}\cdot\text{m}^2$$

Then, $(13.1\ \text{kg}\cdot\text{m}^2)(4.0\ \text{rad/s}) = (5.4\ \text{kg}\cdot\text{m}^2)\omega_f$ or $\omega_f = 9.70$ rad/s.

(*b*)
$$(E_k)_f = \tfrac{1}{2}I_f\omega_f{}^2 = \tfrac{1}{2}(5.4\ \text{kg}\cdot\text{m}^2)(9.70\ \text{rad/s})^2 = 254\ \text{J}$$
$$(E_k)_i = \tfrac{1}{2}I_i\omega_i{}^2 = \tfrac{1}{2}(13.1\ \text{kg}\cdot\text{m}^2)(4.0\ \text{rad/s})^2 = 105\ \text{J}$$

$\Delta E_k = 254\ \text{J} - 105\ \text{J} = 149$ J.

(*c*) From the internal work done by the student in drawing in his arms with the held masses.

Problem 10.26. A wheel, of moment of inertia $I = 5.0$ slug · ft^2 and radius $R = 2.0$ ft, is initially at rest and free to rotate about a frictionless vertical axis through its center. The wheel has horizontal cups attached to the rim which can catch and hold objects thrown at them tangential to the rim (Fig. 10-18). A ball of mass $m = 0.50$ slug is thrown at one of the cups with a velocity $v = 80$ ft/s.

(*a*) Find the angular velocity of the wheel after the ball is caught in the cup.

(*b*) How much thermal energy was lost in the collision?

Solution

(*a*) Angular momentum about the *z* axis (through the center of the wheel) is conserved since there is no external torque about this axis on either the ball (in flight) or the wheel. Since the ball is moving tangentially, the *z* component of its angular momentum before the collision is mvR. Since the

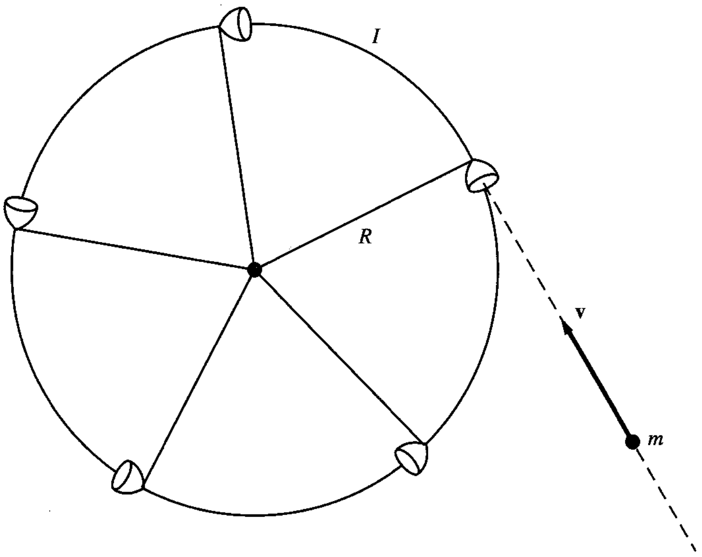

Fig. 10-18

wheel is initially at rest, its angular momentum is zero. The initial z component of angular momentum of the wheel-ball system is therefore

$$L_i = mvR = (0.50\ \text{slug})(80\ \text{ft/s})(2.0\ \text{ft}) = 80\ \text{slug}\cdot\text{ft}^2/\text{s}$$

After the collision, the moment of inertia of the wheel-ball system is

$$I_T = 5.0\ \text{slug}\cdot\text{ft}^2 + (0.50\ \text{slug})(2.0\ \text{ft})^2 = 7.0\ \text{slug}\cdot\text{ft}^2$$

and conservation gives

$$(7.0\ \text{slug}\cdot\text{ft}^2)\omega_f = 80\ \text{slug}\cdot\text{ft}^2/\text{s} \qquad \text{or} \qquad \omega_f = 11.4\ \text{rad/s}$$

(*b*) Thermal heat generated = loss of kinetic energy.

$$(E_k)_i = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.50\ \text{slug})(80\ \text{ft/s})^2 = 1600\ \text{ft}\cdot\text{lb}$$
$$(E_k)_f = \tfrac{1}{2}I_T\omega_f{}^2 = \tfrac{1}{2}(7.0\ \text{slug}\cdot\text{ft}^2)(11.4\ \text{rad/s})^2 = 455\ \text{ft}\cdot\text{lb}$$

Thermal energy = (1600 − 455) ft · lb = 1145 ft · lb.

10.5 ROTATION ABOUT AN AXIS THROUGH THE CENTER OF MASS

Until now we have been discussing rotation about a fixed axis. In general an object can translate through space and rotate at the same time; the dynamics of such motion can be very complicated. The special properties of the CM of a rigid body, however, simplify the analysis. In Fig. 10-19 we show an object moving through space. We imagine a coordinate system moving with the object so that its origin is fixed at the CM of the object, but its axes remain parallel to the axes of a coordinate system fixed in an inertial frame. (The moving coordinate system is called the **CM frame**.) From the point of view of someone moving with the CM frame, the CM is pinned at the origin, and therefore the object's only motion is its rotation about some axis through the origin. If this rotation axis has a fixed direction (does not pivot) as observed in the CM frame—if, for example, it always coincides with the z axis of the frame—then $\Gamma = I\alpha$ holds about this axis in the CM frame, as do the other laws of rotation. This is true even if the CM is accelerating relative to the inertial frame and therefore is not itself an inertial

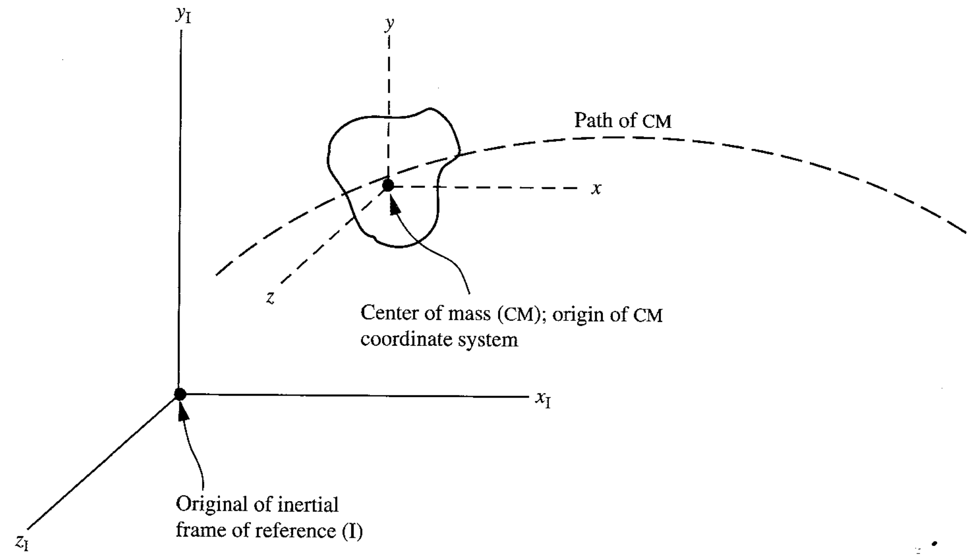

Fig. 10-19

frame. The laws of rotation generally *do not hold* in a coordinate system whose origin is fixed at a point in the body *other* than the CM.

Another special result for the CM frame is that the total kinetic energy of an object, as measured in the inertial frame, is given by

$$E_k = \tfrac{1}{2}MV_{\mathrm{CM}}{}^2 + \tfrac{1}{2}I_{\mathrm{CM}}\omega^2 \tag{10.19}$$

Here, M is the mass of the object, V_{CM} is the velocity of the CM as seen in the inertial frame, I_{CM} is the moment of inertia about the CM axis of rotation, and ω is the angular velocity (which is the same in both coordinate systems, since their axes stay parallel). Note that the first term on the right of *(10.19)* is just the kinetic energy the body would have in the inertial frame if it were not rotating at all but merely translating with the velocity of the CM. The second term is just the rotational kinetic energy of the body as seen in the CM frame, i.e., the total kinetic energy in the CM frame.

Use of the CM frame and the energy decomposition (*10.19*) are particularly valuable in analyzing rolling motion.

Problem 10.27. In Fig. 10-20(*a*) a uniform disk, of mass $M = 2.0$ kg and radius $R = 0.45$ m, rolls without slipping down an inclined plane of length $L = 35$ m and slope 30 °. The disk starts from rest at the top of the incline. Find (*a*) the angular acceleration α of the disk and the linear acceleration a of the CM of the disk; (*b*) the time for the disk to reach the bottom of the incline; (*c*) the angular velocity at the bottom of the incline.

Solution

(*a*) Since there is no slippage between disk and incline, the point of contact translates a distance Δx along the incline when the disk has rotated through an arc length $\Delta s = \Delta x$. If $\Delta\theta$ is the corresponding angle of rotation, we have $\Delta s = R\,\Delta\theta$ and therefore $\Delta x = R\,\Delta\theta$. This leads to $v = R\omega$ and $a = R\alpha$, where ω and α are the angular velocity and angular acceleration of the disk, and where v and a are the linear velocity and acceleration of the point of contact along the incline. Since the CM of the disk is always in the same position relative to the point of contact, v and a are also the velocity and acceleration of the CM.

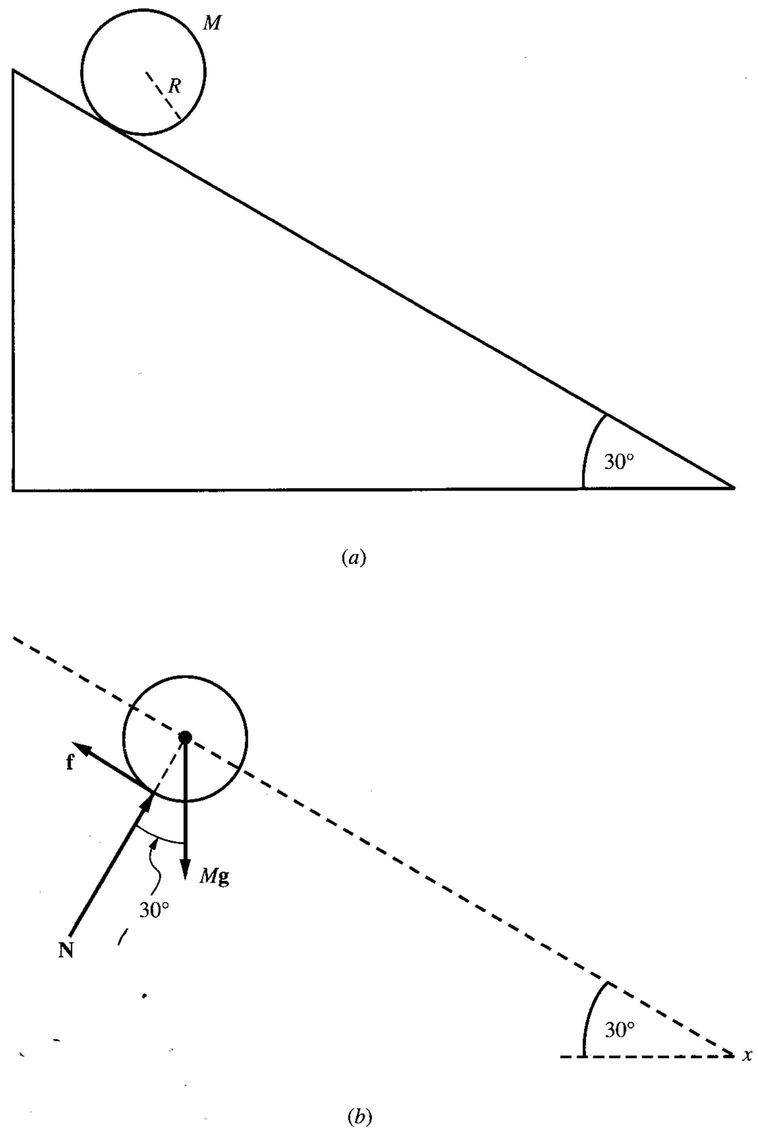

Fig. 10-20

Using the body diagram of the disk, Fig. 10-20(*b*), we have for the *x* motion of the CM

$$Mg \sin 30^\circ - f = Ma \qquad (i)$$

Both f and a are unknowns, so a second equation is needed. For rotation *about the axis through the* CM we have $\Gamma = I_{CM}\alpha$. If we choose clockwise as positive and refer to Fig. 10-11(*a*), this becomes $fR = (\frac{1}{2}MR^2)(a/R)$, or

$$f = \tfrac{1}{2}Ma \qquad (ii)$$

Adding (*i*) and (*ii*) eliminates f and yields $Mg \sin 30^\circ = [M + \frac{1}{2}M]a$, or

$$a = \tfrac{2}{3}g \sin 30^\circ = \frac{9.8 \text{ m/s}^2}{3} = 3.27 \text{ m/s}^2$$

and $\alpha = a/R = (3.27 \text{ m/s}^2)/(0.45 \text{ m}) = 7.27 \text{ rad/s}^2$.

(*b*) Since the disk starts from rest, $L = \frac{1}{2}at^2$, or

$$t = \sqrt{\frac{2L}{a}} = \sqrt{\frac{2(35\ \text{m})}{3.27\ \text{m/s}^2}} = 4.63\ \text{s}$$

(*c*)
$$\omega = \alpha t = (7.27\ \text{rad/s}^2)(4.63\ \text{s}) = 33.7\ \text{rad/s}$$

Problem 10.28. Use conservation of energy to solve Problem 10.27(*c*).

Solution

Since there is no slippage between disk and incline, the frictional force does no work. Similarly, the normal force does no work. Thus, only gravity does work, and mechanical energy is conserved. The initial kinetic energy is zero, so $E_{kf} = \frac{1}{2}Mv^2 + \frac{1}{2}I_{CM}\omega^2$ = loss of potential energy = $MgL \sin 30°$. Substituting in $I_{CM} = \frac{1}{2}MR^2$ and $v = \omega R$, we have $\frac{1}{2}(MR^2 + \frac{1}{2}MR^2)\omega^2 = MgL \sin 30°$. Dividing out M, $\frac{3}{4}R^2\omega^2 = gL \sin 30° \Rightarrow 0.75(0.45\ \text{m})^2\omega^2 = (9.8\ \text{m/s}^2)(35\ \text{m})(0.50) \Rightarrow \omega = 33.6$ rad/s, which agrees with Problem 10.27(*c*) to within rounding errors.

Problems for Review and Mind Stretching

Problem 10.29. A uniform sphere of radius $R = 30$ cm is made of a material of density $\rho = 5000\ \text{kg/m}^3$.

(*a*) Find the moment of inertia about an axis through the center of the sphere.

(*b*) How would the moment of inertia change if the radius doubled and the density stayed the same?

(*c*) How would the moment of inertia of part (*a*) change if the radius doubled while the mass stayed the same?

Solution

(*a*) By Fig. 10-11(*e*),

$$I = \frac{2}{5}MR^2 \qquad (i)$$

Recalling that the volume of a sphere is $V = \frac{4}{3}\pi R^3$, we get $M = \rho V = \frac{4}{3}\pi\rho R^3$. Therefore,

$$I = \frac{8\pi\rho R^5}{15} = \frac{8(3.14)(5000\ \text{kg/m}^3)(0.30\ \text{m})^5}{15} = 20.3\ \text{kg}\cdot\text{m}^2 \qquad (ii)$$

(*b*) From Eq. (*ii*) of part (*a*) we see that for constant ρ, I varies as R^5. Therefore, doubling R increases I by a factor of $2^5 = 32$. Then $I = 32(20.3\ \text{kg}\cdot\text{m}^2) = 650\ \text{kg}\cdot\text{m}^2$.

(*c*) From Eq. (*i*) of part (*a*) we see that for constant M, I varies as R^2. Therefore doubling R increases I by a factor of $2^2 = 4$. Then $I = 4(20.3\ \text{kg}\cdot\text{m}^2) = 81.2\ \text{kg}\cdot\text{m}^2$.

Problem 10.30. In an Atwood's machine (Fig. 10-21) two blocks are connected by a light cord over a pulley of radius R and moment of inertia I. Assume the pulley rotates without friction on a horizontal axis and there is no slippage between the cord and the pulley. Find (*a*) the acceleration of the blocks; (*b*) the tensions T_A and T_B on either side of the pulley.

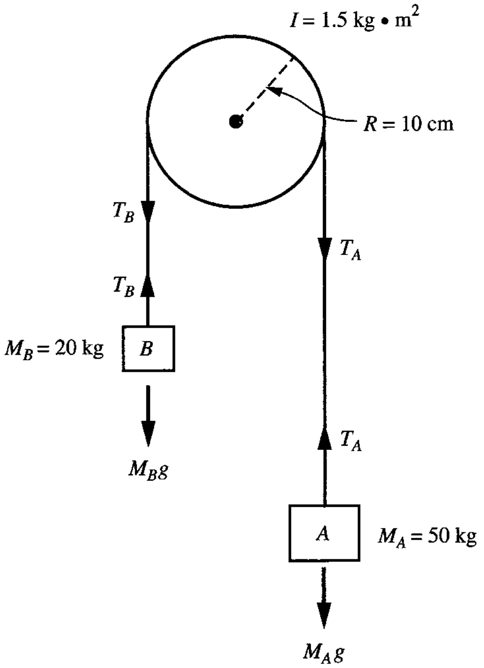

Fig. 10-21

Solution

(*a*) We apply the laws of motion to each of the three objects, for consistency choosing downward as positive for block A, upward as positive for block B, and clockwise as positive for the pulley. Then a can represent the acceleration of either block, and $\alpha = a/R$ represents the angular acceleration of the pulley.

for block A:

$$M_A g - T_A = M_A a \qquad (i)$$

for block B:

$$T_B - M_B g = M_B a \qquad (ii)$$

for the pulley:

$$\Gamma = I\alpha \Rightarrow T_A R - T_B R = \frac{Ia}{R} \quad \text{or} \quad T_A - T_B = \frac{I}{R^2}a \qquad (iii)$$

Adding (*i*), (*ii*), and (*iii*), the tensions cancel and we get

$$(M_A - M_B)g = \left(M_A + M_B + \frac{I}{R^2}\right)a$$

$$(50\ \text{kg} - 20\ \text{kg})(9.8\ \text{m/s}^2) = \left[50\ \text{kg} + 20\ \text{kg} + \frac{1.5\ \text{kg}\cdot\text{m}^2}{(0.10\ \text{m})^2}\right]a$$

$$a = 1.34\ \text{m/s}^2$$

(*b*) From (*i*), $T_A = M_A(g - a) = (50\ \text{kg})\ (9.8\ \text{m/s}^2 - 1.34\ \text{m/s}^2) = 423\ \text{N}$. Similarly, from (*ii*),

$$T_B = M_B(g + a) = (20\ \text{kg})(9.8\ \text{m/s}^2 + 1.34\ \text{m/s}^2) = 223\ \text{N}$$

Problem 10.31. A small block of mass $m = 200$ g is constrained to move in a circle of radius $R_i = 20$ cm on a horizontal frictionless tabletop by a cord that passes through a hole in the center of the table and is held in position, as shown in Fig. 10-22. Initially the block has a velocity of $v_i = 35$ cm/s. The cord is then very slowly pulled down until the radius of the circle of motion drops to one-half its original value.

(*a*) Find the initial angular momentum l_i of the block about a vertical axis through the hole.

(*b*) What is the velocity of the pulled-in block?

(*c*) How much work was done by the force pulling down on the cord?

(*d*) What were the initial and final tensions in the cord?

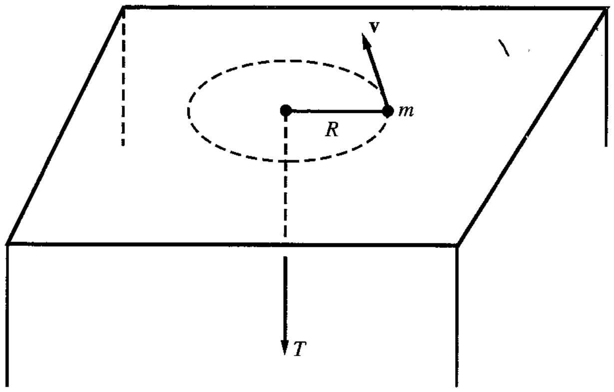

Fig. 10-22

Solution

(*a*) Since the radius is perpendicular to the velocity of the block,

$$l_i = mv_iR_i = (0.200\ \text{kg})(0.35\ \text{m/s})(0.20\ \text{m}) = 0.014\ \text{kg}\cdot\text{m}^2/\text{s}$$

(*b*) The only force in the plane of motion is the tension in the cord. Its line of action passes through the center of the circle and hence contributes zero torque about the vertical axis through that point. Thus, angular momentum about that axis is conserved:

$$mv_fR_f = mv_iR_i \qquad \text{or} \qquad v_f = \frac{R_i}{R_f}v_i = 2(35\ \text{cm/s}) = 70\ \text{cm/s}$$

(*c*) The only work done on the block is by the force pulling the cord down. Applying the work-energy theorem,

$$(E_k)_i = \tfrac{1}{2}mv_i^2 = \tfrac{1}{2}(0.20\ \text{kg})(0.35\ \text{m/s})^2 = 0.0123\ \text{J}$$
$$(E_k)_f = \tfrac{1}{2}mv_f^2 = \tfrac{1}{2}(0.20\ \text{kg})(0.70\ \text{m/s})^2 = 0.0490\ \text{J}$$
$$W = (E_k)_f - (E_k)_i = 0.0367\ \text{J}$$

(*d*) The initial tension is given by the centripetal force law:

$$T_i = \frac{mv_i^2}{R} = \frac{(0.20\ \text{kg})(0.35\ \text{m/s})^2}{0.20\ \text{m}} = 0.123\ \text{N}$$

Similarly, the final tension is given by

$$T_f = \frac{mv_f{}^2}{R} = \frac{(0.20\ \text{kg})(0.70\ \text{m/s})^2}{0.10\ \text{m}} = 0.980\ \text{N}$$

Problem 10.32. Ten girls, of mass $m = 35$ kg each, initially stand at different points on the rim of a merry-go-round; its moment of inertia is $I_0 = 800\ \text{kg}\cdot\text{m}^2$, and its radius is $R = 4.0$ m. The merry-go-round has an initial angular velocity of $\omega_i = 0.20$ rad/s and rotates freely on frictionless bearings.

(*a*) Find the velocity and the centripetal acceleration of each girl.

(*b*) On a signal the girls all start moving in toward the center of the merry-go-round until they reach a distance $R' = 1.5$ m from the center, where they again stand. Find the new angular velocity ω_f of the system.

Solution

(*a*) For each girl, $v = \omega R = (0.20\ \text{rad/s})(4.0\ \text{m}) = 0.80$ m/s, and **v** is tangential. Similarly, for each girl, the centripetal acceleration has magnitude

$$a_R = \omega^2 R = (0.20\ \text{rad/s})^2\ (4.0\ \text{m}) = 0.16\ \text{m/s}^2$$

(*b*) Angular momentum about the axis of rotation is conserved: $I_i\omega_i = I_f\omega_f$. Here,

$$I_i = 10(35\ \text{kg})(4.0\ \text{m})^2 + 800\ \text{kg}\cdot\text{m}^2 = 6400\ \text{kg}\cdot\text{m}^2$$
$$I_f = 10(35\ \text{kg})(1.5\ \text{m})^2 + 800\ \text{kg}\cdot\text{m}^2 = 1588\ \text{kg}\cdot\text{m}^2$$

Thus $$(6400)(0.20\ \text{rad/s}) = 1588\omega_f \qquad \text{or} \qquad \omega_f = 0.806\ \text{rad/s}$$

Problem 10.33. In Problem 10.32, take the merry-go-round by itself as constituting the system. If 3.0 s was required for the girls to move from the rim to the final position, calculate (*a*) the time-average power and (*b*) the time-average torque exerted by the girls on the merry-go-round. (*c*) Assuming the torque was actually constant over the time interval, use it to find P_{av}.

Solution

(*a*) The average power P_{av} is given by the work-energy theorem as P_{av} = work/time = $\Delta E_k/\Delta t$, where ΔE_k is the change in kinetic energy of the merry-go-round. From Problem 10.32,

$$\Delta E_k = \tfrac{1}{2}I_0\omega_f{}^2 - \tfrac{1}{2}I_0\omega_i{}^2 = (400\ \text{kg}\cdot\text{m}^2)[(0.806\ \text{rad/s})^2 - (0.200\ \text{rad/s})^2] = 244\ \text{J}$$

Then,

$$P_{av} = 244\ \text{J}/3.0\ \text{s} = 81.3\ \text{W}$$

(*b*) The rotational impulse-momentum theorem gives $\Gamma_{av}\,\Delta t = I_0(\omega_f - \omega_i)$, or

$$\Gamma_{av} = \frac{(800\ \text{kg}\cdot\text{m}^2)(0.806\ \text{rad/s} - 0.200\ \text{rad/s})}{3.0\ \text{s}} = 162\ \text{N}\cdot\text{m}$$

(*c*) $P_{av} = \Gamma\omega_{av}$. If Γ is constant, then so is the angular acceleration α, and $\omega_{av} = (\omega_f + \omega_i)/2 =$ (0.806 rad/s + 0.200 rad/s)/2 or $\omega_{av} = 0.503$ rad/s. Then $P_{av} = (162\ \text{N}\cdot\text{m})(0.503\ \text{rad/s}) =$ 81.5 W, which is consistent with our result in part (*a*).

Supplementary Problems

Problem 10.34. A wheel having a uniform angular acceleration starts from rest and rotates through 150 rad in 2.5 s. Find (*a*) the angular acceleration; (*b*) the angular velocity at the end of the 2.5-s interval; (*c*) the number of revolutions turned through in the 2.5 s.

Ans. (*a*) 48 rad/s^2; (*b*) 120 rad/s; (*c*) 23.9 r

Problem 10.35. The wheel of Problem 10.34 has a radius of 30 cm. At $t = 2.5$ s, find, for any point on the rim, (*a*) the linear velocity, (*b*) the tangential acceleration, and (*c*) the centripetal acceleration.

Ans. (*a*) 36 m/s; (*b*) 14.4 m/s^2; (*c*) 4320 m/s^2

Problem 10.36. Referring to Problem 10.34, suppose that at the end of the 2.5-s interval the angular acceleration is suddenly changed to -144 rad/s^2. Find (*a*) the angular velocity 2.5 s later; (*b*) the overall angular displacement in the full 5.0-s interval.

Ans. (*a*) -240 rad/s; (*b*) 0 rad

Problem 10.37. A rigid body in the *xy* plane consists of three identical spokes making angles of 120 ° with one another [Fig. 10-23]. Each spoke is of length $L = 2.0$ m and mass $m = 3.0$ kg and has a bob of mass $M = 5.0$ kg attached at the far end. Find the moment of inertia of the rigid body about the perpendicular (*z*) axis through its center. [*Hint*: See Fig. 10-11.]

Ans. 72 kg · m^2

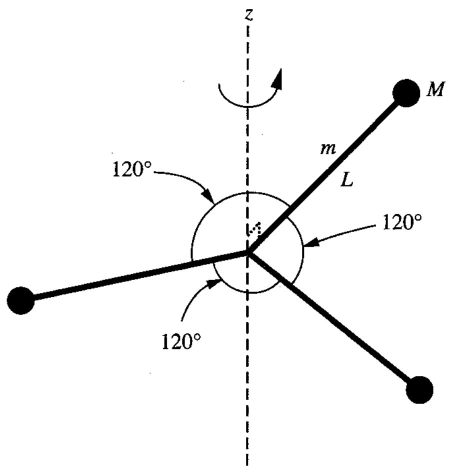

Fig. 10-23

Problem 10.38. A force of constant magnitude $F = 12$ N is applied to one of the bobs in the rigid body of Problem 10.37. The force acts in the plane of the rigid body and always points at right angles to the spoke to which the bob is attached. If the rigid body is free to rotate about the *z* axis, and starts from rest, find (*a*) the torque about the rotation axis, (*b*) the angular velocity when the object has rotated through 80 rad.

Ans. (*a*) 24 N · m; (*b*) 7.30 rad/s

Problem 10.39. A block of mass $M = 15$ kg starts from rest on a frictionless inclined plane of angle 40 °. The block is attached to a cord whose other end is wrapped around a wheel of moment of inertia $I = 15\ \text{kg} \cdot \text{m}^2$ and radius $R = 80$ cm, as shown in Fig. 10-24. The wheel is free to rotate about a frictionless horizontal axle. Find (*a*) the acceleration of the block down the incline; (*b*) the tension in the cord.

Ans. (*a*) 2.46 m/s^2; (*b*) 57.6 N

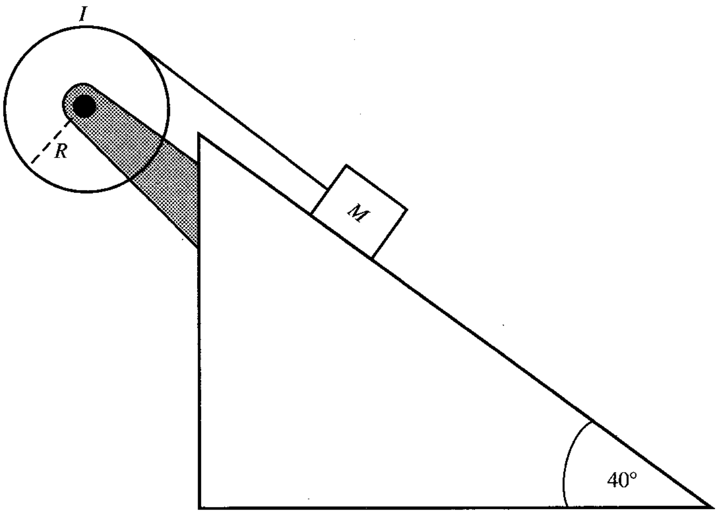

Fig. 10-24

Problem 10.40. In Problem 10.39, use energy considerations to find the angular velocity of the wheel after the block has moved 3.0 m down the incline.

Ans. 4.80 rad/s

Problem 10.41. Referring to Problem 10.40, suppose that there were friction between the block and the incline. If the angular velocity of the wheel turned out to be 3.80 rad/s when the block moved the 3 m along the incline, (*a*) how much thermal energy was produced in the process? (*b*) What is the coefficient of kinetic friction between the block and the incline?

Ans. (*a*) 106 J; (*b*) 0.313

Problem 10.42. A grinding wheel of radius $R = 20$ cm is rigidly connected to a long horizontal axle that is supported on both ends by frictionless pivots. The axle is driven by a belt over a pulley, which is also rigidly connected to the axle; the situation is shown in Fig. 10-25. A tool is pressed perpendicular to the grinding wheel with a force $N = 50$ N. Assume that μ_k between tool and wheel is 0.60 and that the wheel rotates at a steady rate of 10 r/s.

(*a*) Find the torque on the grinding wheel, about the axis of rotation.
(*b*) What must be the torque exerted by the belt on the wheel (or pulley)?
(*c*) Assuming no losses in the shaft-belt-pulley system, how much power is supplied by the motor?

Ans. (*a*) 6.0 N · m; (*b*) −6.0 N · m; (*c*) 377 W

Problem 10.43. Suppose that the moment of inertia of the wheel, shaft, and pulley of Problem 10.42 is 30 kg · m^2. Assume the same conditions as in Problem 10.42, and that the belt suddenly breaks. If the tool

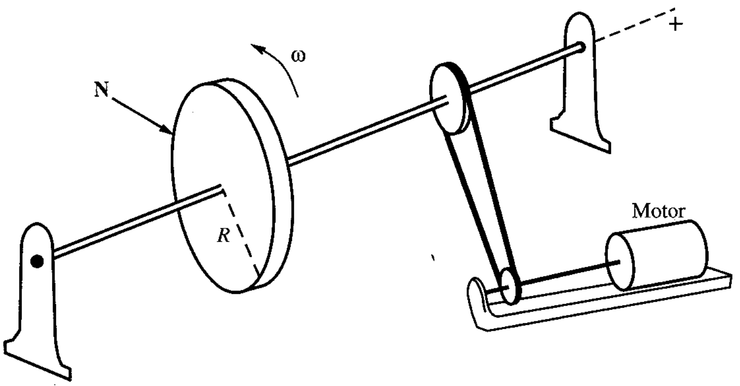

Fig. 10-25

remains pressed against the grinding wheel, (*a*) how long will it take for the grinding wheel to come to a stop? (*b*) how much thermal energy will be generated in that time period?

Ans. (*a*) 314 s; (*b*) 59.2 kJ

Problem 10.44. A uniform rod of mass $M = 6.0$ kg and length $L = 2.6$ m is free to swing in a vertical circle about a pivot at one end (point A), as shown in Fig. 10-26. If the rod is released from rest from the position shown, (*a*) find its angular velocity as it coincides with the negative y-axis. [*Hint*: Use energy conservation.] (*b*) Is angular momentum conserved in this motion?

Ans. (*a*) 4.12 rad/s; (*b*) no

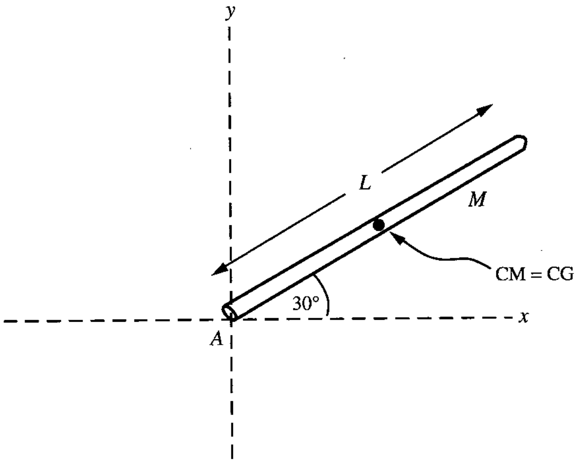

Fig. 10-26

Problem 10.45. A lump of clay, of mass $m = 0.20$ slug, is dropped from a height of 10.0 ft directly above one end of a uniform horizontal rod of length $L = 4.0$ ft and mass $M = 3.0$ slugs. The rod is free to rotate in a vertical circle about a horizontal axis through its center A (Fig. 10-27). Just before the collision, what is (*a*) the velocity of the clay, (*b*) the angular momentum of the clay (about the axis through point A)?

Ans. (*a*) 25.4 ft/s; (*b*) 10.2 slug · ft^2/s

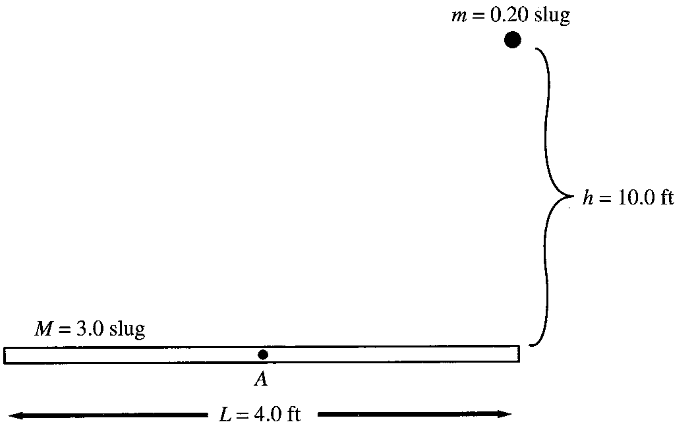

Fig. 10-27

Problem 10.46. Assume that the collision of Problem 10.45 is very rapid and that the clay sticks to the rod. Find (*a*) the angular velocity and (*b*) the kinetic energy, of the rod–clay combination immediately after the collision. (*c*) How much thermal energy is generated in the collision process?

Ans. (*a*) 2.11 rad/s; (*b*) 10.7 ft · lb; (*c*) 53.7 ft · lb

Problem 10.47. Refer to Problem 10.24 and Fig. 10-16. Assume that the initial height difference between the two disks is 1.5 m and that the mass of the upper disk is $M_A = 100$ kg.

(*a*) How much of the initial rotational kinetic energy is lost in the collision of the two disks?

(*b*) How much thermal energy is generated?

Ans. (*a*) 20.2 kJ; (*b*) 21.7 kJ

Problem 10.48. The mass of the earth is $M_e = 5.98 \times 10^{24}$ kg and its radius is $R_e = 6380$ km. Assuming the earth is a perfect sphere, find its angular momentum and rotational kinetic energy about its axis of rotation.

Ans. 7.08×10^{33} kg · m²/s, 2.57×10^{29} J

Problem 10.49. Referring to Problem 10.48, imagine that an asteroid of mass $m = 5.0 \times 10^6$ kg, appears in the plane of the earth's equator. Traveling in a westerly direction at $v = 10$ km/s, the asteroid hits the earth just tangent to its surface, sticking to it.

(*a*) What is the angular momentum of the meteor about the earth's axis before the collision?

(*b*) Will this collision change the earth's angular momentum by a significant amount?

(*c*) Assuming the same speed, what would the mass of the asteroid have to be to bring the rotation of the earth to a halt?

Ans. (*a*) 3.19×10^{17} kg · m²/s; (*b*) no (the change is insignificant; (*c*) 1.11×10^{23} kg (i.e., more massive than the moon)

Problem 10.50. A boy of mass $m = 30$ kg stands on the rim of a merry-go-round of moment of inertia $I = 200$ kg · m² and radius $R = 2.0$ m that is free to rotate without friction about a vertical axis through its center. Initially the angular velocity of the system is $\omega = 3.0$ rad/s. Suddenly the boy starts running around the rim in the direction *opposite* to the rotation, with a velocity relative to the earth of $v = 10$ m/s.

(*a*) What is the angular velocity of the merry-go-round while the boy is running?
(*b*) How fast does the boy seem to be running to someone at rest on the rim?

Ans. (*a*) 7.8 rad/s; (*b*) 25.6 m/s

Problem 10.51. Redo Problem 10.27(*a*) if the disk is replaced by (*a*) a thin hoop and (*b*) a sphere of the same mass and radius.

Ans. (*a*) $a = 2.45\ \text{m/s}^2$, $\alpha = 5.44\ \text{rad/s}^2$; (*b*) $a = 3.50\ \text{m/s}^2$, $\alpha = 7.78\ \text{rad/s}^2$

Problem 10.52. How would the results of Problems 10.27(*a*) and 10.51 change if the mass were different?

Ans. There would be no change, since the mass drops out of the equations.

Problem 10.53. Write general expressions—in terms of mass M, radius R, and center-of-mass velocity V_{CM}—for the kinetic energies of the following uniform objects that are rolling without slipping: (*a*) a disk; (*b*) a hoop; (*c*) a sphere.

Ans. (*a*) $\frac{3}{4}MV_{\text{CM}}^2$; (*b*) MV_{CM}^2; (*c*) $\frac{7}{10}MV_{\text{CM}}^2$