Chapter 1

Introduction and Mathematical Background

1.1 INTRODUCTION TO THE STUDY OF PHYSICS AND ITS RELATIONSHIP TO MATHEMATICS

What Is Physics?

Physics is the study of all physical phenomena—phenomena experienced through the human senses, either directly or with the aid of instruments. Among the topics studied are the following: (*a*) the motion and distortion of objects due to interaction with their environment (mechanics); (*b*) heat and thermodynamics; (*c*) sound and other wave motions; (*d*) light and optical phenomena; (*e*) electrical and magnetic phenomena; and (*f*) atomic and molecular properties of matter. The typical general physics sequence, consisting of two or three courses, usually covers all these subjects at an elementary level, often in the order listed.

Physics Is a Science

It is based on experiment and observation. It is a *quantitative* science; relationships between physical quantities (such as position and time for a moving object) are expressed as precisely as possible. That is why physics uses the language of mathematics. Only mathematical formulas can give the relationships between physical quantities in precise form. Thus, for example, if we wish to describe what happens to an object that is dropped from a certain height, we can do so with different degrees of precision. At the lowest level of precision we can say that the object falls until it hits the ground. At a somewhat more detailed level we can say that the object speeds up as it falls. At the most detailed level we want to know exactly where it is located and how fast it is moving at every instant of time. For this last case we need a mathematical relationship between the height and the time and between the speed of fall and the time.

1.2 MATHEMATICAL REVIEW

Notation

We will often use a center dot ($\cdot$) to signify the product of two numbers and a slash (/) to signify their division. When no ambiguity can exist, the center dot for multiplication will be omitted [for example, when multiplying terms in parentheses: 3(10) to be read "three times ten" or when multiplying by a variable: $2x$ to be read "two times x"]. The **absolute value** or **magnitude of a** number, or of a variable x, is its value with a positive sign. Using the notation "absolute value of x" = $|x|$, we have, for example: $|-6| = 6$; $|3| = 3$; for x negative, $x = -|x|$, while for x positive, $x = |x|$.

Mathematical Functions

Suppose we have a mathematical relationship between two quantities (for example, the relationship between the distance that an object moves and the time that has elapsed or the relationship between two purely mathematical quantities). Then, if one of the quantities takes on a particular value, the relationship tells us the corresponding value of the other quantity. Suppose the two quantities are represented by the symbols x and t; then for every value of t there is a definite value of x. The quantities x and t are called **variables** because they can take on a range of values. The variable x is said to be a **function** of the variable t, or symbolically, $x = f(t)$, which is read "x equals a function of t," or in shorthand, x equals "eff" of t.

Problem 1.1. The quantity x is related to the quantity t by the relationship $x = f(t) = 3t + 4$. Find the value of x when t takes on each of the following values: $t = 0$, $t = 1$, $t = 2$, t = 2.5, $t = 10$.

Solution

For the first value we have

$$x = f(0) = 3 \cdot 0 + 4 = 0 + 4 = 4$$

Similarly, for the other values we have

$$x = f(1) = 3 \cdot 1 + 4 = 3 + 4 = 7 \qquad x = f(2) = 3 \cdot 2 + 4 = 6 + 4 = 10$$

$$x = f(2.5) = 3(2.5) + 4 = 7.5 + 4 = 11.5 \qquad x = f(10) = 3 \cdot 10 + 4 = 30 + 4 = 34$$

Problem 1.2. The quantities z and x are related by the function $z = f(x) = 12x - 7$. Find the values of z corresponding to the following values of x: $x = 3, -3, \frac{1}{2}, -\frac{1}{2}, 1.2$.

Solution

We carry out the calculation for each value of x:

$$z = f(3) = 12 \cdot 3 - 7 = 29 \qquad z = f(-3) = 12(-3) - 7 = -43$$

$$z = f\left(\tfrac{1}{2}\right) = 12\left(\tfrac{1}{2}\right) - 7 = -1 \qquad z = f\left(-\tfrac{1}{2}\right) = 12\left(-\tfrac{1}{2}\right) - 7 = -13$$

$$z = f(1.2) = 12(1.2) - 7 = 14.4 - 7 = 7.4$$

Problem 1.3. The quantities y and x are related through the equation $y = f(x) = 2x^2$. Find y for $x = -4, -2, -1, 0, 1, 2, 4, 4.5$.

Solution

We calculate y for each value of x, in the order given. To save space we won't write the $f(x)$ expression for each case.

$$y = 2(-4)^2 = 2 \cdot 16 = 32 \qquad y = 2(-2)^2 = 2 \cdot 4 = 8$$

$$y = 2(-1)^2 = 2 \qquad y = 2(0)^2 = 0 \qquad y = 2(1)^2 = 2$$

$$y = 2(2)^2 = 8 \qquad y = 2(4)^2 = 32 \qquad y = 2(4.5)^2 = 2 \cdot 20.25 = 40.5$$

Problem 1.4. y and t are related by the function $y = f(t) = 4t^2 - 2t + 6$. Find y when $t = -3, -2, -1, 0, 1, 2, 3$.

Solution

In the order given,

$$y = 4(-3)^2 - 2(-3) + 6 = 36 + 6 + 6 = 48$$
$$y = 4(-2)^2 - 2(-2) + 6 = 16 + 4 + 6 = 26$$
$$y = 4(-1)^2 - 2(-1) + 6 = 4 + 2 + 6 = 12$$
$$y = 4(0)^2 - 2(0) + 6 = 0 + 0 + 6 = 6$$
$$y = 4(1)^2 - 2(1) + 6 = 4 - 2 + 6 = 8$$
$$y = 4(2)^2 - 2(2) + 6 = 16 - 4 + 6 = 18$$
$$y = 4(3)^2 - 2(3) + 6 = 36 - 6 + 6 = 36$$

Graphs

Whenever one has a mathematical relationship between two variables, say x and t, one can represent the function $x = f(t)$ by a two-dimensional graph. To do this, one typically draws two straight lines, called **axes** at right angles to each other—one horizontal and the other vertical, as shown in Fig. 1-1. The horizontal axis is marked off to some scale, as shown, for the variable t (called the **independent variable**) with negative and positive values as shown. The zero point, where the two axes cross, is called the **origin** and is denoted by the letter O for origin or the numeral 0. Traditionally the right half of the horizontal axis is chosen as positive and the left half as negative. Similarly, the values of x (called the **dependent variable**) are marked off on some scale on the vertical axis, with upward traditionally chosen as positive.

For each value of t on the horizontal axis, one imagines a line drawn vertically upward to a point (point A in Fig. 1-1, for example) whose height corresponds to the value of $x = f(t)$ as measured on

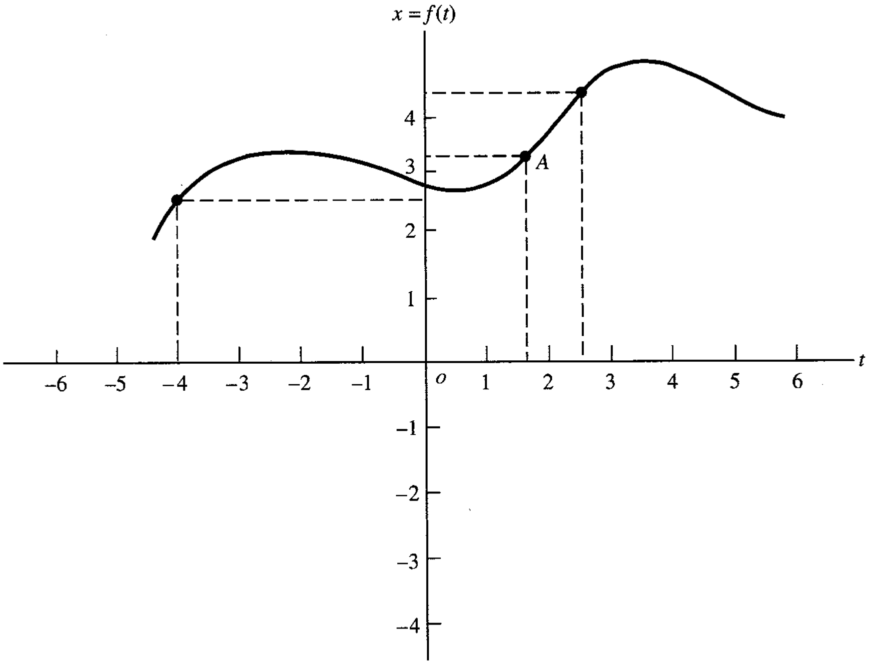


Fig. 1-1

the vertical axis. One can imagine a horizontal line from the vertical axis also drawn to the point A. (The two imaginary lines are represented by dashed lines in Fig. 1-1). If one constructs points this way for each value of t (two more examples are shown in the figure), the points can be fitted by a smooth curve as shown. This curve is the **graph** of the function $x = f(t)$. Each point on the graph is directly above (or directly below, for negative x) a value of t. The same point is then directly to the right (or to the left) of the corresponding value of x. Thus all the information contained in the relationship between x and t [that is, the function $x = f(t)$] is displayed on the graph.

Problem 1.5. Plot the graph of the function in Problem 1.1 between the values $t = -5$ and $t = +5$.

Solution

We first plot some of the values already calculated in Problem 1.1. These are shown in Fig. 1-2. As can be seen they lie on a straight line. The x intercept, defined as the point at which the line crosses the vertical axis, is at $x = 4$. The **slope** of the line, defined as the number of vertical units between two points on the line divided by the corresponding number of horizontal units, is just $\frac{6}{2} = 3$, as demonstrated in the figure by means of the dashed lines. (Since the scale of our graph was chosen differently for the horizontal and the vertical axes, the vertical distance doesn't *look* three times as big as the horizontal distance.)

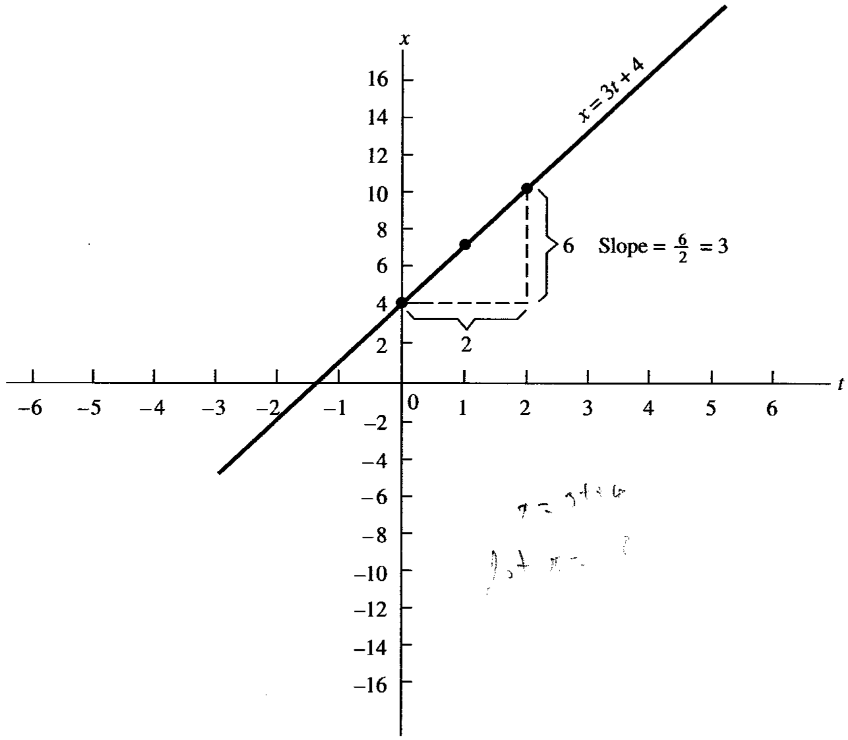


Fig. 1-2

A straight line is always specified uniquely on a graph by giving the slope and the intercept with the vertical axis. In fact the general equation of a straight line with slope m and intercept b is just $x = mt + b$.

Problem 1.6. Plot the function $z = f(x)$ found in Problem 1.2.

Solution

The function is $z = 12x - 7$. We plot x on the horizontal and z on the vertical and see that this is clearly a straight line whose vertical intercept is at $z = -7$ and whose slope is 12. We have to plot only two points to draw the straight line, and this is illustrated in Fig. 1-3.

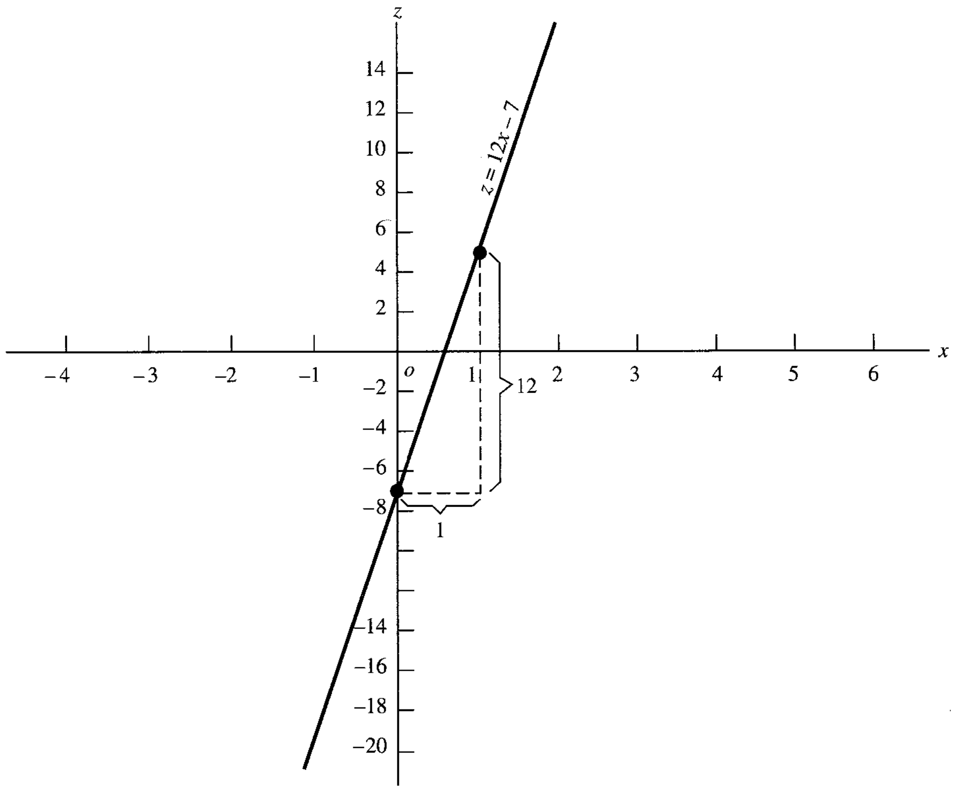


Fig. 1-3

Problem 1.7. Plot the function $y = f(x)$ found in Problem 1.3.

Solution

Since $y = 2x^2$, this is clearly *not* a straight line. Figure 1-4 shows the graph of this function; note some of the values calculated in Problem 1.3. This is the curve of a parabola that is symmetric about the vertical axis and touches the origin at its lowest point.

Problem 1.8. Plot the function $x = f(t)$ where $f(t)$ is given in Problem 1.4.

Solution

Here the function is $x = 4t^2 - 2t + 6$ and it is shown plotted in Fig. 1-5. Again we have a parabola, but now it is symmetric about a vertical axis through the point $t = \frac{1}{4}$, which corresponds to its lowest point $x = 5.75$.

Inverse Functions

When we have a function $x = f(t)$, for every t value we can determine the corresponding x value. Is it possible to turn this around so that for every x value we can find a corresponding t value? The

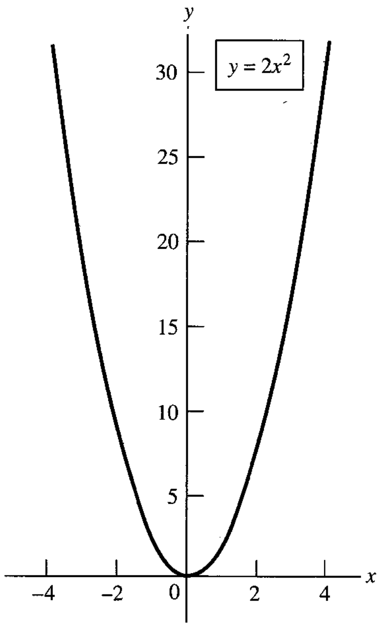


Fig. 1-4

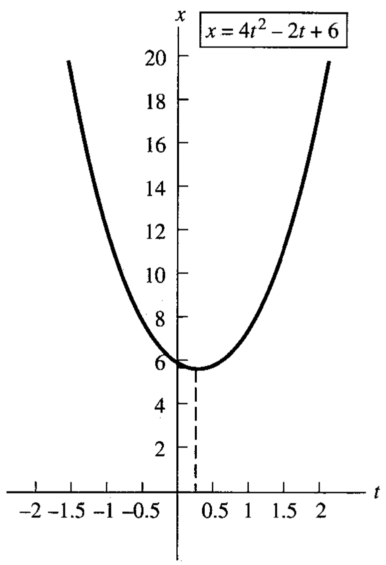


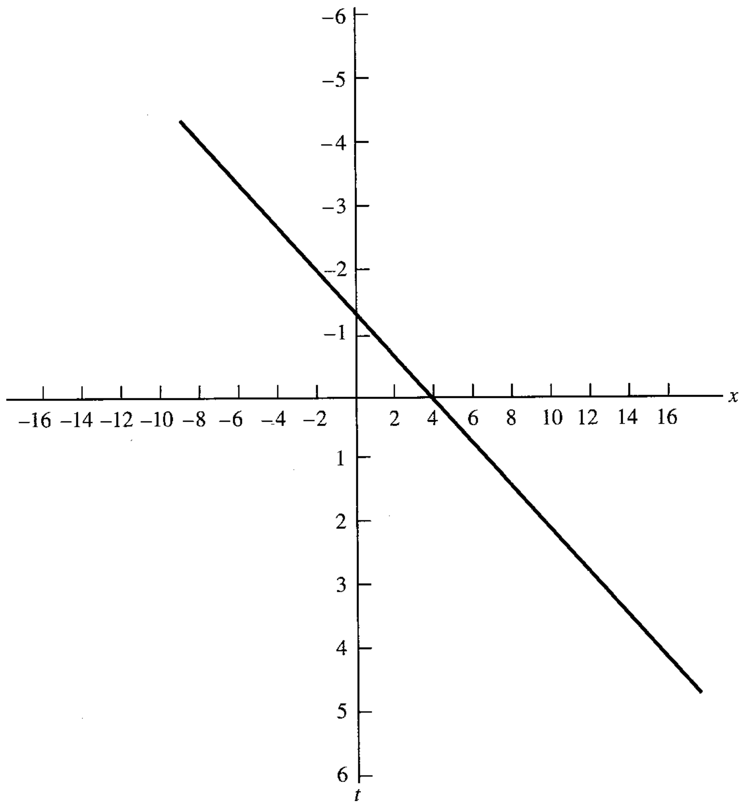
Fig. 1-5

answer is a qualified yes. Sometimes there is no ambiguity, but at other times one must exercise caution. If we succeed in doing so, then t has become a function of x. To acknowledge that this new function was obtained by turning around or "inverting" the function $x = f(t)$, it is usually labeled $t = f^{-1}(x)$ and is called the **inverse function**.

Problem 1.9. Find the inverse function for the case of Problem 1.1.

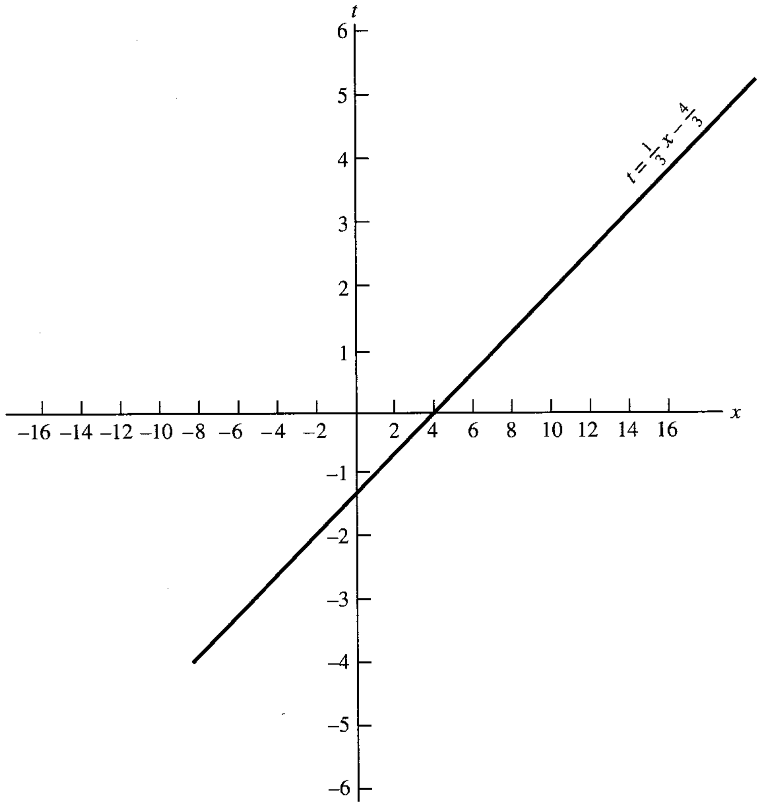
Solution

The original function $x = f(t)$ is given by the equation $x = 3t + 4$. To get t in terms of x we want to isolate the t in the equation. First we subtract 4 from both sides of the equation, to get $3t = x - 4$. Next



−6
−5
−4
−3
−2
−1
−16 −14 −12 −10 −8 −6 −4 −2 2 4 6 8 10 12 14 16
x
1
2
3
4
5
6
t

(a)



t
6
5
4
3
2
1
−16 −14 −12 −10 −8 −6 −4 −2 2 4 6 8 10 12 14 16
x
−1
−2
−3
−4
−5
−6
$t = \frac{1}{3}x - \frac{4}{3}$

(b)

Fig. 1-6

we divide both sides by 3, to get $t = (x - 4)/3$. Finally, simplifying this last result, we get for our inverse function, $t = f^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$. This is the equation of a straight line of slope $\frac{1}{3}$ and t intercept $(-\frac{4}{3})$.

Problem 1.10 shows how the graph of the inverse function can be easily obtained from that of the original function, with no further calculation.

Problem 1.10. Obtain the graph of the inverse function determined in Problem 1.9.

Solution

The graph of the original function is shown in Fig. 1.2. Rotate that figure 90° clockwise so that x appears along the horizontal [Fig. 1-6(a)]. Then t appears along the vertical, except the negative values are up and the positive are down. This can be corrected by flipping the t axis 180° about the x axis [Fig. 1-6(b)]. Thus the inverse function is just the same graph rotated so that the dependent and independent variables change place.

Problem 1.11. Find the inverse function for the function given in Problem 1.3.

Solution

Here we have the quadratic function $y = 2x^2$. Again we try to isolate x. First we divide both sides of the equation by 2 to get $x^2 = y/2$. Then we take the square root of both sides of the equation to get $x = \pm\sqrt{y/2}$. In this case, because of the plus and minus signs, we really have two different inverse functions:

$$\text{(i)}\quad x = +\sqrt{\frac{y}{2}} \qquad \text{(ii)}\quad x = -\sqrt{\frac{y}{2}}$$

This can be understood by inverting the graph of the original function (Fig. 1-4). As in Problem 1.10, we rotate by 90° clockwise and then flip by 180° about the new horizontal axis to get the graph for the inverse function shown in Fig. 1-7. Notice that there are now two values of x, one positive and one negative, for each value of y. Each branch of the curve, above and below the axis, corresponds to the choice of (i) or (ii) above, respectively, as the inverse function. Another interesting feature is that the inverse function (either branch) is not defined for all values of y, but rather only for positive values of y (see Fig. 1-7). It is often the case that a function $y = f(x)$ is defined for all x, but the inverse function $x = f^{-1}(y)$ is defined for only a limited range of y values.

Trigonometric Functions

Among the mathematical functions that are particularly important in a general physics course are the trigonometric functions. The most commonly used trigonometric functions are the sine, cosine,

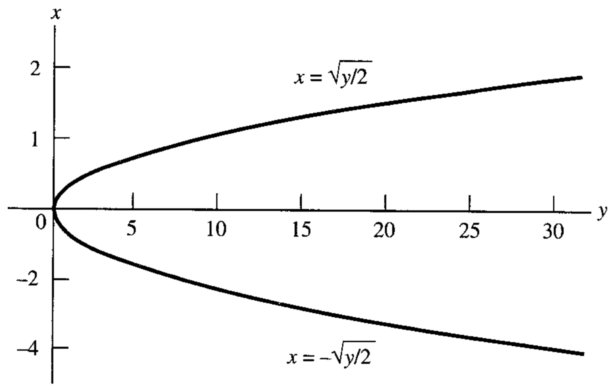


Fig. 1-7

and tangent functions. They are most usually defined in terms of ratios of sides of a right triangle, in which an angle (say, θ) plays the role of the independent variable. Figure 1-8 shows the rules for obtaining the sine, cosine, and tangent of angles between $\theta = 0°$ and $\theta = 360°$. The angle is always measured counterclockwise from the positive horizontal axis to the hypotenuse of the right triangle.

$$\begin{aligned}
&\textbf{sine of } \theta \qquad && \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{o}{h} \\
&\textbf{cosine of } \theta \qquad && \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{h} \\
&\textbf{tangent of } \theta \qquad && \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{o}{a}
\end{aligned} \tag{1.1}$$

Also, from (*1.1*)

$$\tan\theta = \frac{o/h}{a/h} = \frac{\sin\theta}{\cos\theta} \tag{1.2}$$

The trigonometric functions are positive or negative depending on the quadrant. The correct signs ($\pm$) for the functions in all four quadrants can be determined by using Fig. 1-8. The rule is that the opposite and adjacent sides of the triangles shown are to be considered positive or negative depending on which side of the axis they are on, while the hypotenuses are always considered positive. (Note that only in the first quadrant is the angle of interest θ inside the triangle.) The graphs of the trigonometric functions are shown in Fig. 1-9(*a*), (*b*), and (*c*), where we have plotted $x = \sin\theta$, $x = \cos\theta$, and

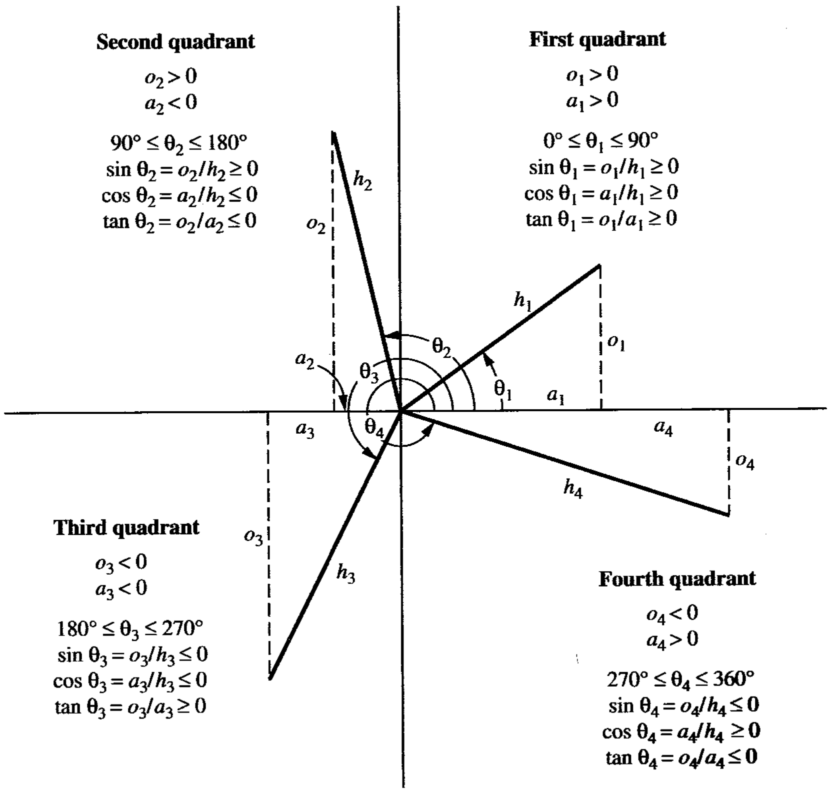


Fig. 1-8

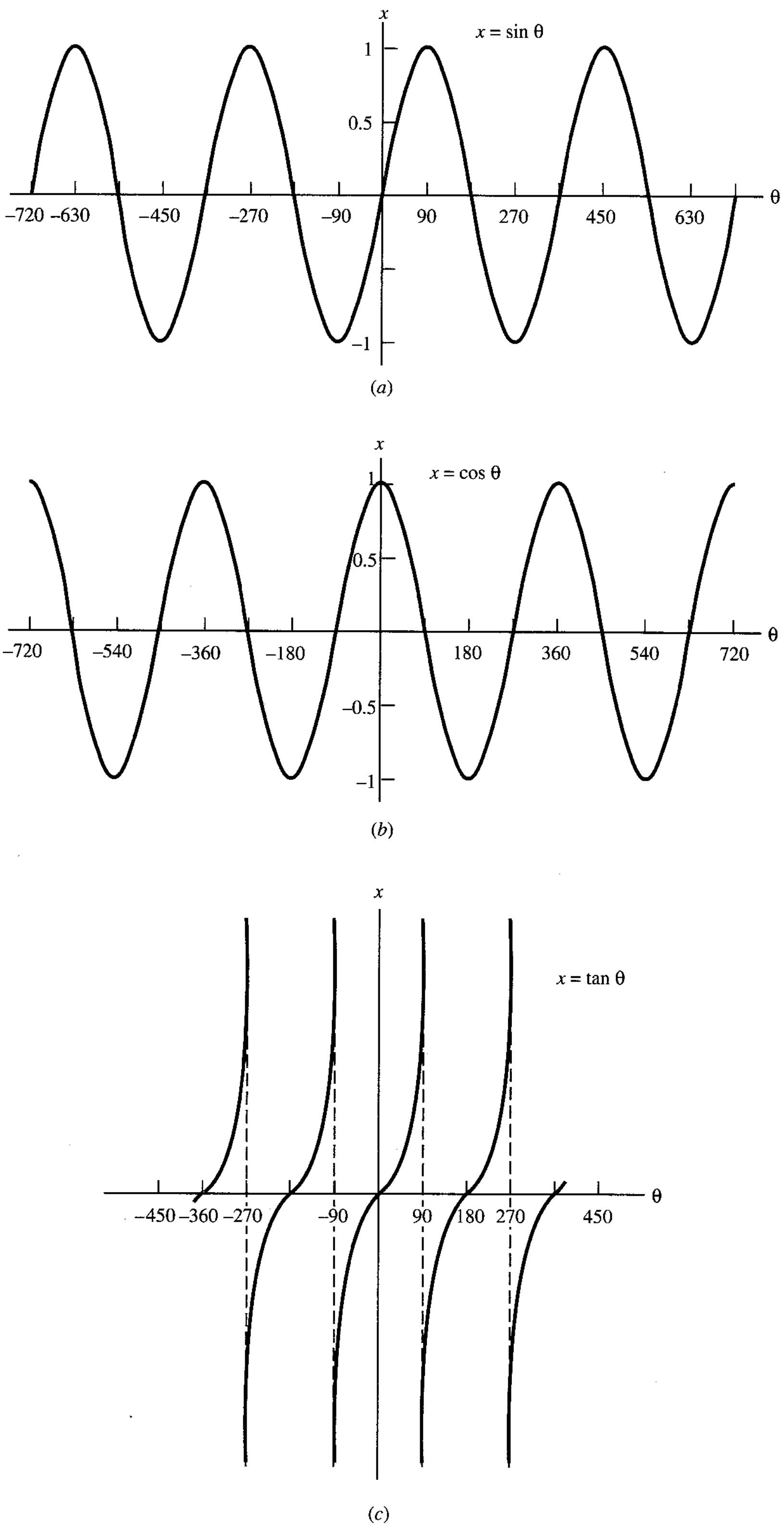

x
x = sin θ
1
0.5
−1
θ
−720 −630
−450
−270
−90
90
270
450
630
(a)
x
x = cos θ
1
0.5
−0.5
−1
θ
−720
−540
−360
−180
180
360
540
720
(b)
x
x = tan θ
θ
−450 −360 −270
−90
90
180 270
450
(c)

Fig. 1-9

$x = \tan\theta$, respectively. It is seen that the functions repeat themselves every time θ winds through $360°$, clockwise or counterclockwise:

$$\begin{aligned} \sin\theta &= \sin(\theta + 360°) = \sin(\theta + 720°) = \cdots \\ \cos\theta &= \cos(\theta + 360°) = \cos(\theta + 720°) = \cdots \\ \tan\theta &= \tan(\theta + 360°) = \tan(\theta + 720°) = \cdots \end{aligned} \tag{1.3}$$

Thus, for example, if $\theta = 300°$, we have

$$\sin(300°) = \sin(-60°) \qquad \cos(300°) = \cos(-60°) \qquad \text{and} \qquad \tan(300°) = \tan(-60°) \tag{1.4}$$

Figure 1-9 indicates that the maximum and minimum values of the sine and cosine functions are ± 1. This is a consequence of the fact that the length of the sides a and o can never exceed that of the hypotenuse. The tangent, however, can vary from minus infinity to plus infinity, since $\tan\theta = \sin\theta/\cos\theta$ and the expression becomes infinite when the cosine becomes zero.

Problem 1.12. What is the sign of the sine, cosine, and tangent in each quadrant?

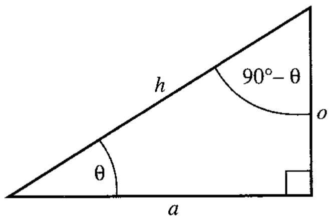
Solution

From Fig. 1-8, using the sign convention discussed, we see that in the first quadrant (where θ is an acute angle) o, a, and h are all positive, so all three functions are positive. In the second quadrant (where θ is between $90°$ and $180°$), o and h are positive, but a is negative. Thus, $\sin\theta$ is positive, while $\cos\theta$ and $\tan\theta$ are negative. In the third quadrant (where θ is between $180°$ and $270°$), both o and a are negative, and only h is positive. Thus, $\sin\theta$ and $\cos\theta$ are both negative, while $\tan\theta$ is positive. In the fourth quadrant (where θ is between $270°$ and $360°$), o is negative, while a and h are positive. Thus, $\sin\theta$ and $\tan\theta$ are negative, while $\cos\theta$ is positive.

Problem 1.13. Show that $\sin\theta = \cos(90° - \theta)$; $\cos\theta = \sin(90° - \theta)$; $\tan(90° - \theta) = \cot\theta$ (where $\cot\theta$ is defined as $\cos\theta/\sin\theta$, and thus $\cot\theta = 1/\tan\theta$).

Solution

Consider the right triangle in Fig. 1-10. Since opposite and adjacent sides for angle θ are adjacent and opposite sides, respectively, for angle $(90° - \theta)$, we get all three results.



$$\sin\theta = \frac{o}{h} = \cos(90° - \theta)$$

$$\cos\theta = \frac{a}{h} = \sin(90° - \theta)$$

$$\tan\theta = \frac{1}{\tan(90° - \theta)}$$

Fig. 1-10

Problem 1.14. Show that (*a*) $\cos\theta = -\cos(180° - \theta)$; (*b*) $\sin\theta = \sin(180° - \theta)$.

Solution

(*a*) We consider the second-quadrant triangle depicted in Fig. 1-11(*a*). For angle θ we must consider o and h positive and a negative. Then $a = -|a|$ and $\cos\theta = -|a|/h$. On the other hand, $(180° - \theta)$ is an acute angle in a triangle with positive sides o, $|a|$, and h (dotted triangle in first quadrant). Therefore $\cos(180° - \theta) = |a|/h$. Comparing, we get the result $\cos\theta = -\cos(180° - \theta)$.

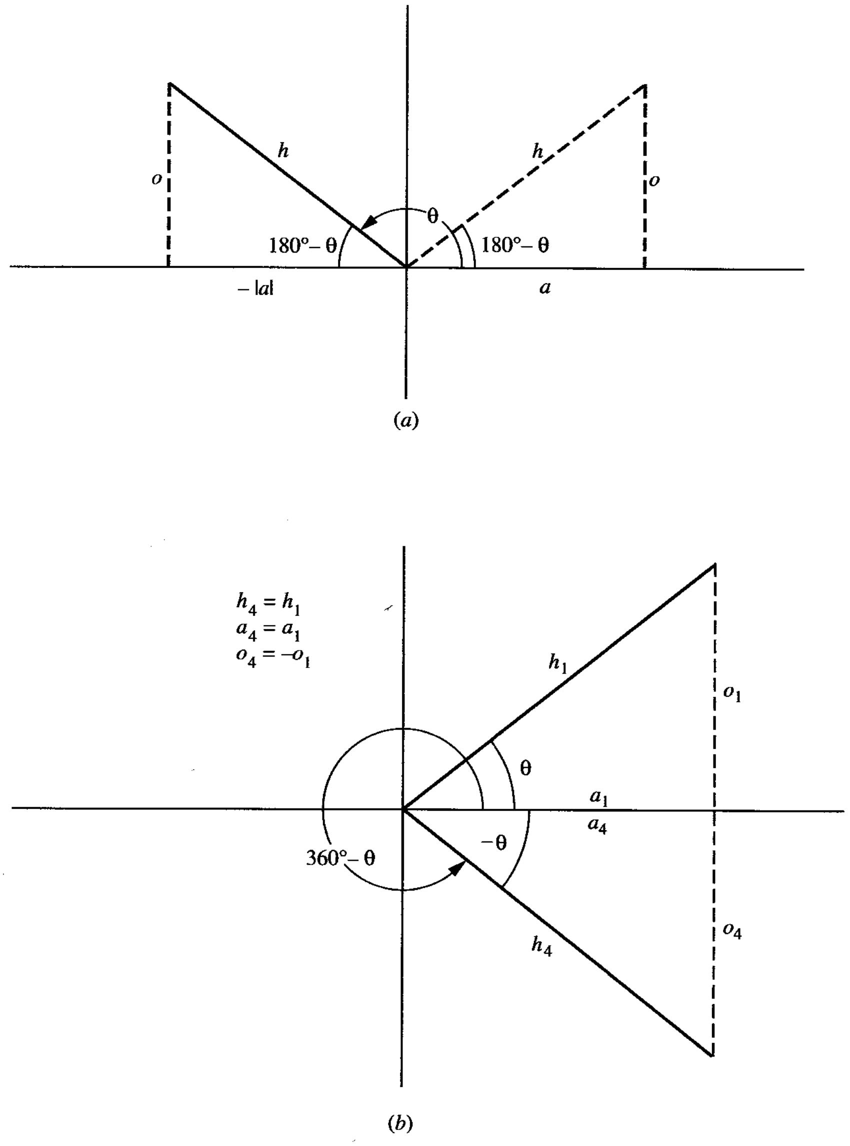


Fig. 1-11

(*b*) Since side o is positive for both θ and $(180° - \theta)$, the sine of each angle equals o/h, and they are equal.

Problem 1.15. Show that $\sin^2\theta + \cos^2\theta = 1$ for all values of θ.

Note. $\sin^2\theta$ stands for $(\sin\theta)^2$, etc., and is the accepted way of writing the square of a trigonometric function.

Solution

$\sin^2\theta + \cos^2\theta = (o/h)^2 + (a/h)^2 = (o^2 + a^2)/h^2$. But, by the pythagorean theorem, $o^2 + a^2 = h^2$, or $(o^2 + a^2)/h^2 = 1$.

Problem 1.16. Show that $\cos(-\theta) = \cos\theta$; $\sin(-\theta) = -\sin\theta$; $\tan(-\theta) = -\tan\theta$.

Solution

By examining the graphs in Fig. 1-9, we see that the cosine is symmetric about the $\theta = 0$ mark, while the sine and tangent are antisymmetric (i.e., if they assume a value at a given positive angle, they will assume the exact negative of that value at the corresponding negative angle). This is precisely what had to be shown. The same results follow directly from the definitions (Fig. 1-8). Consider an angle θ in the first quadrant and the corresponding angle $-\theta$ below the horizontal in the fourth quadrant, as shown in Fig. 1-11(*b*). Angle $(360 - \theta)$ is the counterclockwise angle to the hypotenuse in the fourth quadrant, so the trigonometric functions for $(-\theta)$ can be obtained from the triangle shown in the fourth quadrant with the usual sign convention. The two triangles shown are congruent, so the sides a, o, and h have the same magnitudes in both quadrants, but side o has opposite signs in the two quadrants. Thus, since sine and tangent involve side o, we have $\sin\theta = -\sin(-\theta)$, $\tan\theta = -\tan(-\theta)$. For the cosine, which does not involve side o, $\cos\theta = \cos(-\theta)$.

In general, to find the sine, cosine, or tangent of a particular angle, one has to use trigonometric tables or a calculator. However, for certain angles that often come up in general physics problems, one can obtain the values of the trigonometric functions from Fig. 1-12. The triangle in Fig. 1-12(*a*) is referred to as a "30-60-90" degree triangle; Fig. 1-12(*b*) shows the isosceles right triangle; and Fig. 1-12(*c*) is the "3-4-5" sides triangle. (In this last triangle, the angles given are not exact, but they are a good approximation.)

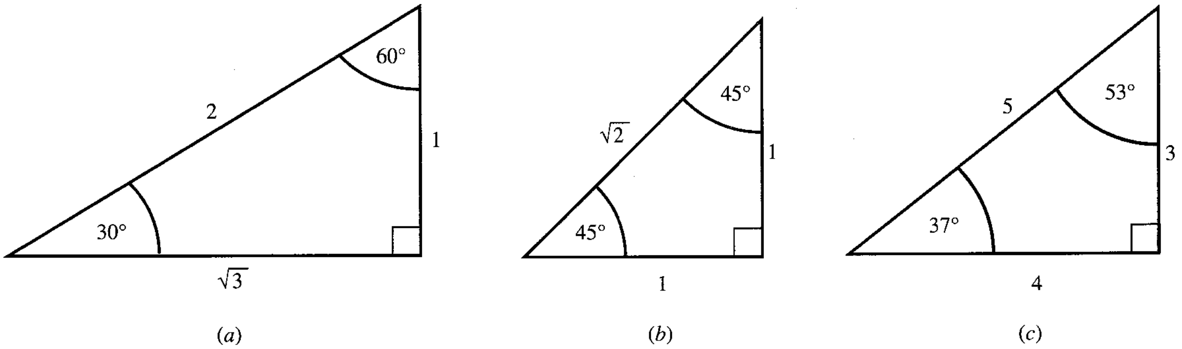


Fig. 1-12

Problem 1.17.

(*a*) Find the following from Fig. 1-12: $\sin 30°$, $\cos 30°$, $\tan 60°$, $\cos 45°$, $\sin 37°$, $\cos 53°$.

(*b*) Find the following values of the inverse trigonometric functions: $\sin^{-1}(\frac{1}{2})$, $\cos^{-1}(\sqrt{3}/2)$, and $\tan^{-1}(1)$.

Solution

(*a*) From Fig. 1-12(*a*) and Eqs. (*1.1*): $\sin 30° = o/h = \frac{1}{2} = 0.500$; $\cos 30° = a/h = \sqrt{3}/2 = 0.866$; $\tan 60° = o/a = \sqrt{3}/1 = 1.73$. From Fig. 1-12(*b*): $\cos 45° = a/h = 1/\sqrt{2} = 0.707$. From Fig. 1-12(*c*): $\sin 37° = o/h = \frac{3}{5} = 0.600$; $\cos 53° = a/h = \frac{3}{5} = 0.600$.

(*b*) From the definition of inverse functions: $x = \sin\theta \Leftrightarrow \theta = \sin^{-1} x$. So $\theta = \sin^{-1}(\frac{1}{2}) \Rightarrow \sin\theta = \frac{1}{2}$. From Fig. 1-12(*a*) [or part (*a*)] we have $\theta = 30°$. However, since the sine is also positive in the second quadrant, we get a second solution: $\theta = 180° - 30° = 150°$. Similarly, $\theta = \cos^{-1}(\sqrt{3}/2) \Rightarrow \cos\theta = \sqrt{3}/2 \Rightarrow \theta = 30°$. Since cosine is also positive in the fourth quadrant, we have a second solution: $\theta = 360° - 30° = 330°$. Repeating for the next case: $\theta = \tan^{-1} 1 \Rightarrow \tan\theta = 1$. From Fig. 1-12(*b*): $\tan 45° = \frac{1}{1} = 1$, so $\theta = 45°$. Tangent is also positive in the third quadrant, so we have another solution: $\theta = 180° + 45° = 225°$.

Actually there are an infinite number of solutions to part (*b*) if we include angles outside the range from $0°$ to $360°$, as can be seen from Eqs. (*1.3*), or by turning Figs. 1-9 (*a*), (*b*), (*c*) on their sides as in Problems 1.10 and 1.11. However, since the angles repeat every $360°$, it is usually sufficient to consider only angles in the range $0°$ to $360°$.

Problem 1.18. Figure 1-13 shows a ramp of length L and angle θ, whose base is $b = 6.25$ m and whose height is $h = 2.15$ m. Find (*a*), θ, (*b*) L.

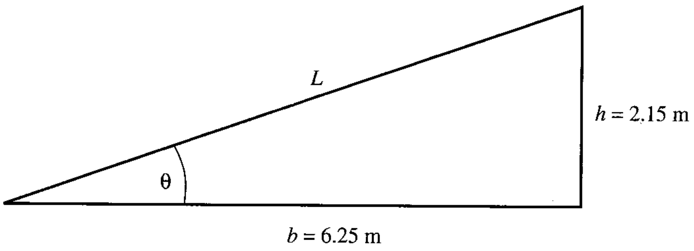


Fig. 1-13

Solution

(*a*) $\theta = \tan^{-1}(h/b) = \tan^{-1}(2.15/6.25) = \tan^{-1}(0.344)$. Using a calculator or tables, we get $\theta = 19.0°$, ignoring the solution in the third quadrant because we know θ is acute.

(*b*) We can get L from the fact that $\sin\theta = h/L$, so $L = h/\sin\theta$ or $L = 2.15\text{ m}/\sin 19.0° = 2.15\text{ m}/0.326 = 6.60$ m. This result may be checked by use of the pythagorean theorem:

$$L^2 = b^2 + h^2 = (6.25\text{ m})^2 + (2.15\text{ m})^2 = 43.7\text{ m}^2 \qquad \text{or} \qquad L = 6.61\text{ m}$$

which checks within rounding errors.

Simultaneous Equations

Often in solving physics problems one encounters two relationships involving the same two variables. Then both relationships can be valid only for specific values of the variables. To see this we look at Figure 1-14, which shows the graphs of two functions between the variables y and x: $y = f_1(x)$ and $y = f_2(x)$. As can be seen, for an arbitrary value of x the corresponding y values will be different for the two functions. There is, however, one particular value of x (call it x_A) for which the two functions give the same value of y (call it y_A). This is the only pair of values of x and y that are valid for (or "satisfy") both relationships. We say that the two "simultaneous" equations between y and x (given by the two functions) are "solved" by the values x_A and y_A. (If the curves cross in more than one place, there exist additional pairs of values that satisfy both relationships, so there are additional solutions to the pair of equations.)

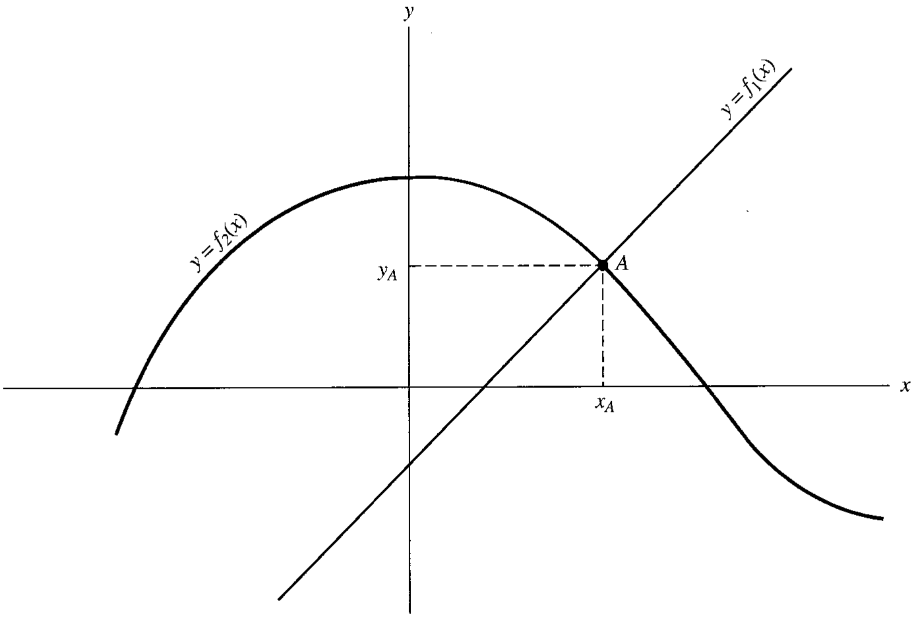


Fig. 1-14

Sometimes the two relationships can be expressed in the particularly simple form

$$a_1x + b_1y = c_1 \qquad \text{and} \qquad a_2x + b_2y = c_2 \tag{1.5a, b}$$

where a_1, b_1, c_1 and a_2, b_2, c_2 are constants, and x, y are the variables. (A specific example would be

$$6x + 3y = 8 \qquad \text{and} \qquad 2x - 3y = 4 \tag{1.6a, b}$$

Here $a_1 = 6$, $b_1 = 3$, $c_1 = 7$, $a_2 = 2$, $b_2 = -3$, $c_2 = 4$.) Such equations are represented by straight lines on a graph, since one can rewrite them in the standard slope and intercept form

$$y = mx + b \tag{1.7}$$

(See Problem 1.19.) Equations of the form (*1.5*) are therefore called **linear equations**. Since two straight lines can cross at most once, there is either a single solution to the pair of equations or no solution when they don't cross at all.

Problem 1.19. Show that Eq. (*1.6a*), $6x + 3y = 8$, is a straight line on a y vs. x graph.

Solution

Subtracting $6x$ from both sides of the equation we get $3y = -6x + 8$. Then we divide both sides by 3 to get $y = -2x + \frac{8}{3}$, which is the equation of a straight line expressed in standard form. It has slope $m = -2$ and vertical intercept $b = \frac{8}{3}$.

Problem 1.20. Find the solution of the pair of Eqs. (*1.6a*) and (*1.6b*).

Solution

The two equations are

$$\text{(i)} \quad 6x + 3y = 8 \qquad \text{and} \qquad \text{(ii)} \quad 2x - 3y = 4$$

We must try to get an equation that involves only one of the variables so that it can be solved for the value of the variable. In Problem 1.19 we showed that (i) can be rewritten as

$$\text{(iii)}\quad y = -2x + \tfrac{8}{3}$$

We substitute this expression for y into (ii), obtaining

$$\text{(iv)}\quad 2x - 3\left(-2x + \tfrac{8}{3}\right) = 4$$

Performing the multiplication by -3 we get

$$\text{(v)}\quad 2x + 6x - 8 = 4 \qquad \text{or} \qquad 8x = 12$$

which yields $x = \frac{12}{8} = \frac{3}{2} = 1.5$. Substituting back into (iii), we obtain $y = -2(1.5) + \frac{8}{3} = -3 + \frac{8}{3} = -\frac{1}{3}$. As a check of our results we substitute our values for x and y back into (ii) and get $2(1.5) - 3(-\frac{1}{3}) = 3 + 1 = 4$, as required.

Problem 1.21. Solve the simultaneous equations

$$\text{(i)}\quad z = 3t + 4 \qquad\qquad \text{(ii)}\quad z = 12t - 7$$

Except for the labeling of the variables, these are the straight-line equations of Problems 1.1 and 1.2 above.

Solution

The two expressions for z must be equal, or $3t + 4 = 12t - 7$. Bringing the terms with t to one side of the equation and the constant terms to the other yields $9t = 11$ or $t = \frac{11}{9}$. Then, substituting the value of t into, say (i), we get $z = 3(\frac{11}{9}) + 4 = \frac{11}{3} + 4 = 7\frac{2}{3}$.

Problem 1.22. Solve the simultaneous equations

$$\text{(i)}\quad 3z + 4t = -2 \qquad\qquad \text{(ii)}\quad 2z - 12t = 3$$

Solution

We could solve these equations by the techniques of Problems 1.20 and 1.21, but let us use another method instead. We note that if we multiply both sides of (i) by 3 we get

$$\text{(iii)}\quad 9z + 12t = -6$$

So the coefficient of t has the same magnitude but opposite sign in both equations. We now add the left sides of (ii) and (iii), so the variable t cancels out. The result must equal the addition of the right sides of the two equations, and we get $11z = -3$ or $z = -\frac{3}{11}$. Substituting the value of z back into (i) yields

$$3\left(-\tfrac{3}{11}\right) + 4t = -2 \qquad \text{or} \qquad 4t = -2 + \tfrac{9}{11} = -\tfrac{13}{11} \qquad \text{or} \qquad t = -\tfrac{13}{44}$$

[Check these results for z and t by showing they satisfy (ii).]

Problem 1.23. Solve the simultaneous equations

$$\text{(i)}\quad y = 2x^2 \qquad\qquad \text{(ii)}\quad 2x + y = 12$$

Solution

Here only one of the equations is linear, the other is a quadratic. We can still solve for x and y, as follows: First we isolate y on one side of (ii): $y = -2x + 12$. Then we substitute this for y in (i) to get

$$\text{(iii)}\quad 2x^2 = -2x + 12 \qquad \text{or} \qquad \text{(iv)}\quad 2x^2 + 2x - 12 = 0$$

This is the well-known quadratic equation, of the general form

$$\text{(v)} \quad ax^2 + bx + c = 0$$

where a, b, and c are constants. Its solution is given by

$$\text{(vi)} \quad x = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

For our case $a = 2$, $b = 2$, and $c = -12$, so

$$x = \frac{-2 \pm (4 + 96)^{1/2}}{4} = \frac{-2 \pm \sqrt{100}}{4} = \frac{-2 \pm 10}{4}$$

Thus $x_1 = 2$ and $x_2 = -3$ are the two solutions for x. As luck would have it, (iv), which can be reduced to $x^2 + x - 6 = 0$, can be factored into $(x + 3)(x - 2) = 0$, yielding the two values of x directly. We now find the corresponding y values by substituting each x value into either (i) or (ii). Using (i) we get

$$y_1 = 2x_1{}^2 = 2(2)^2 = 8 \qquad y_2 = 2x_2{}^2 = 2(-3)^2 = 18$$

1.3 MEASURING PHYSICAL QUANTITIES

Measurement and Units

To find precise relationships that describe physical phenomena we must be able to measure physical quantities such as length, area, volume, velocity, acceleration, mass, time, and temperature. To do this we need **units of measurement** for all the quantities we are interested in. For example, the most commonly used unit of time is the *second*, that of length is the *meter* or the *foot*, and that of mass is the *kilogram*. Not all measurable quantities require their own units. Often the unit is automatically defined in terms of other units. For instance, if we have a unit of length, say the meter, then we already have units of area and volume: the square meter (a square 1 meter on a side) and the cubic meter (a cube 1 meter on a side), respectively. Another example is the unit of velocity, the meter per second, which is already defined in terms of units of length and time. Such units are called **derived units**. It turns out that in the subject of mechanics (motion and distortion of objects), only three physical quantities must have their units defined independently. These three quantities are usually taken to be length, mass, and time, and their units are called **fundamental units**.

Standards

To define a fundamental unit, everyone must agree to pick some physical example of the quantity to be measured and say that by definition it corresponds to one unit. Thus, for example, the unit of mass, the kilogram, is defined as a mass equal to that of a particular platinum-iridium cylinder housed in Sèvres, near Paris, France. That physical specimen is called the *standard* which defines the unit. The unit of length, the meter, used to be defined as the length of a particular platinum-iridium bar, but it has since been redefined in terms of the length of a certain wavelength of light, and most recently in terms of the distance light travels in a certain time interval. The standard for time has always been defined in terms of some repetitive phenomenon, such as the spin of the earth on its axis, the revolution of the earth about the sun, and most recently in terms of the oscillations of an atomic clock. The idea of the standard is that everyone can check their measuring apparatus against the standard for accuracy. In the case of time our everyday clocks can always be checked against the *standard* to make sure that they tell the right time. Thus, the most reliable and reproducible object or process makes the best standard.

History of Units

Not too long ago, units for the same physical quantities were defined independently in different countries and were all based on different standards. Today we still have different units used in different countries, but they are now all based on the same standards. This was essential to avoiding confusion and discrepancies in comparing measurements made in different parts of the world. The set of units most commonly used throughout the world, and which is almost exclusively used in scientific work, is called the *International System of units*, and abbreviated SI, from the French name, Système International d'Unités. In mechanics the units are the meter (m), the kilogram (kg), and the second (s), and are what is commonly called the *metric or mks (meter-kilogram-second) system.*

The units of the metric system are multiplied and subdivided by powers of 10 into commonly used subunits. Examples are the **gram** (g), which is one-thousandth of a kilogram; the **nanometer** (nm), which is one-billionth of a meter; the **centimeter** (cm), which is one-hundredth of a meter; the **kilometer** (km), which is one thousand meters; and the **millisecond** (ms), which is one-thousandth of a second. Indeed, the prefixes to the basic unit indicate what power of 10 to multiply or divide by.

In addition to the SI units, another common set of units used in the United States and a few other countries is referred to as the *English System*. Here the fundamental units are length, time, and force, which are respectively the **foot** (ft), the **second** (sec or s), and the **pound** (lb). The foot is now defined to be precisely 0.3048 meter, the second is the same in all systems, and the pound is defined in terms of the weight of a certain mass (given in kilograms) at a certain location on the earth's surface. (The relation between *weight* and *mass* will be discussed in Chap. 5.)

Systems of Units

All measurements involve specifying a multiplicative number and the associated unit as in, for example, "the length of the table is 10 ft" or "the car traveled at 30 m/s." One must be especially careful in relating different measured quantities. For example, if the length of a table is 10 ft and an extension of length 3 m is added on, what is the combined length? Clearly we cannot simply add the two numbers: 10 ft + 3 m = 13? Before adding we must either convert 10 ft to the equivalent length in meters or convert 3 m to the corresponding length in feet.

It is usual to use a consistent set of units when dealing with a given problem. Thus if we use the mks system, it means not only that the fundamental units are the meter, kilogram, and second but also that all the derived units are based on these three. Thus, the unit of velocity is the meter per second (m/s) and the unit of force is the newton (N) (this unit will be discussed in Chap. 5). This assures us that all mathematical equations will be consistent.

Units as Algebraic Quantities

Whenever we multiply or divide physical quantities, we have to figure out what happens to the units. For example, we know that for something moving at constant speed, *distance* = *speed* × *time.* Suppose the speed is 50 feet per second (50 ft/s). To find the distance traveled in 10 s we multiply the speed by the time to get: distance = (50 ft/s) (10 s) = 500 ft.

Notice that the units were treated as algebraic quantities; the seconds canceled out in the numerator and denominator, giving the result in feet.

Consider a conversion from one unit to another. Suppose we are told that a certain backyard is 30 ft long, and we want to express the length in meters. We multiply the length in feet by the number of meters per foot (m/ft) to get the length in meters: (30 ft) (0.3048 m/ft) = 9.144 m.

Problem 1.24. A certain task takes 12 min to accomplish. Find the time it takes in seconds; in hours.

Solution

To get the time in seconds we multiply the time in minutes by the number of seconds per minute: $t = (12\ \cancel{\text{min}})\ (60\ \text{s}/\cancel{\text{min}}) = 720$ s. To get the time in hours we multiply by the number of hours per minute: $t = (12\ \cancel{\text{min}})\ [(\frac{1}{60})\ \text{h}/\cancel{\text{min}}] = 0.20$ h. Equivalently, we could divide the time in minutes by the number of minutes per hour: $t = (12\ \text{min})/(60\ \text{min/h})$. Separating the numerical and unit parts: $t = (\frac{12}{60})$ $[\text{min}/(\text{min/h})] = 0.20(\cancel{\text{min}})\ (\text{h}/\cancel{\text{min}}) = 0.20$ h, as before.

Problem 1.25. Convert the speed $v = 60$ miles per hour to ft/s.

Solution

$v = 60$ mi/h. We must change both the length unit and the time unit to convert this to ft/s. Recalling that there are 5280 feet in a mile (5280 ft/mi) and 3600 seconds in an hour (3600 s/h), we calculate as follows:

$$v = \frac{(60\ \text{mi/h})\ (5280\ \text{ft/mi})}{3600\ \text{s/h}} = \frac{60 \cdot 5280}{3600} (\cancel{\text{mi}}/\cancel{\text{h}})\ (\text{ft}/\cancel{\text{mi}})\ (\cancel{\text{h}}/\text{s}) = 88\ \text{ft/s}$$

Problem 1.26. The dimensions of a rectangular block are width $w = 0.10$ m, length $l = 0.20$ m, and height $h = 0.30$ m. Find the volume in cubic centimeters (cm^3).

Solution

Method 1. We convert each dimension to cm: $w = (0.10\ \cancel{\text{m}})\ (100\ \text{cm}/\cancel{\text{m}}) = 10$ cm, $l = (0.20\ \cancel{\text{m}})$ $(100\ \text{cm}/\cancel{\text{m}}) = 20$ cm, $h = (0.30\ \cancel{\text{m}})\ (100\ \text{cm}/\cancel{\text{m}}) = 30$ cm. Then the volume $v = w \cdot l \cdot h = (10\ \text{cm})$ $(20\ \text{cm})\ (30\ \text{cm}) = 6000\ \text{cm}^3$.

Method 2. We first get the volume in cubic meters (m^3): $v = (0.10\ \text{m})\ (0.20\ \text{m})$ $(0.30\ \text{m}) = 0.0060\ \text{m}^3$. Next we determine how many cubic centimeters (cm^3) there are in a cubic meter (m^3). We note that 1 m = 100 cm, so $(1\ \text{m})^3 = (100\ \text{cm})^3 = (100\ \text{cm})\ (100\ \text{cm})\ (100\ \text{cm})$, or $1\ \text{m}^3 = 1{,}000{,}000\ \text{cm}^3$. Thus the conversion factor is $1{,}000{,}000\ \text{cm}^3/\text{m}^3$. Then $v = (0.0060\ \cancel{\text{m}^3})$ $(1{,}000{,}000\ \text{cm}^3/\cancel{\text{m}^3}) = 6000\ \text{cm}^3$.

It turns out that **units can be treated algebraically** in any physics equation; provided the units are consistent, they will combine to give the correct final unit. As an example of a more complicated situation, let us take the equation describing how far an object travels under constant acceleration (acceleration will be discussed in Chapter 2). The equation is

$$x = v_0 t + \tfrac{1}{2}at^2 \tag{1.8}$$

where v_0 is the velocity at the starting time ($t = 0$ s), a is the constant acceleration, t is the time elapsed in seconds, and x represents how far the object has moved in the elapsed time.

Problem 1.27. In Eq. (*1.8*) we are given that $v_0 = 20$ m/s and that the acceleration corresponds to an increase in velocity of 3 m/s every second. Thus, $a = (3\ \text{m/s})/\text{s} = 3\ \text{m/s}^2$. Find x when $t = 10$ s.

Solution

To find x we substitute all the known information into Eq. (*1.8*), getting

$$x = (20\,\text{m/s})\ (10\,\text{s}) + \tfrac{1}{2}(3\ \text{m/s}^2)\ (10\ \text{s})^2$$

According to our general rule, seconds cancel out of the numerator and denominator in the first expression on the right and $(\text{seconds})^2$ cancel out of the numerator and denominator of the second expression on the right:

$$x = (20\ \text{m}/\cancel{\text{s}})\,(10\ \cancel{\text{s}}) + \tfrac{1}{2}(3\ \text{m}/\cancel{\text{s}^2})\,(100\ \cancel{\text{s}^2}) = 200\ \text{m} + 150\ \text{m} = 350\ \text{m}$$

Always include units in your manipulations, and carry the algebra through on the units. This helps you to keep track of the units and indicates an error in your work if the units don't come out right.

Problem 1.28. In (*1.8*) find x, in meters, if $v_0 = 50$ ft/s, $a = 200\ \text{cm/s}^2$, and $t = 1.5$ min.

Solution

We can't just enter the information into the equation, since the units are not consistent. Since we want x in meters, we convert all distances to meters

$$v_0 = (50\ \cancel{\text{ft}}/\text{s})\,(0.3048\ \text{m}/\cancel{\text{ft}}) = 15.24\ \text{m/s} \qquad a = (200\ \cancel{\text{cm}}/\text{s}^2)\,(0.01\ \text{m}/\cancel{\text{cm}}) = 2.00\ \text{m/s}^2.$$

Similarly we convert minutes to seconds: $t = (1.5\ \cancel{\text{min}})\,(60\ \text{s}/\cancel{\text{min}}) = 90$ s. Now (*1.8*) gives

$$x = (15.24\ \text{m/s})\,(90\ \text{s}) + \tfrac{1}{2}(2.00\ \text{m/s}^2)\,(90\ \text{s})^2 = 1372\ \text{m} + 8100\ \text{m} = 9472\ \text{m}.$$

Significant Figures

Whenever a measured value is given for a physical quantity, it can only be an approximation because it is not possible to measure anything with "infinite" accuracy. For example, in measuring the length of a table with a meterstick one is limited to the accuracy of the rule lines on the stick [see Fig. 1-15(*a*)]. Even if the meterstick were absolutely accurate (an impossibility), one would still have to estimate the fraction of the smallest interval etched on the stick [see Fig. 1-15(*b*)]. Even if the person had the "superhuman" ability to read an "infinitely" accurate ruler, there would still be uncertainty since the apparently smooth edge of the table has some irregularities [Fig. 1-15(*c*)].

A scientist or engineer who specifies the numerical value of a physical quantity keeps only as many figures in the number as are justified by the accuracy to which the physical quantity is known. Thus, if in measuring the length of a table one uses a good meterstick with centimeter gradations etched on it and estimates the fraction of an interval between centimeter marks, one gives the measured value to a tenth of a centimeter, say $L = 3.427$ m. The 7 represents an estimate that may be off by one- or two-tenths of a centimeter. In other words specifying that $L = 3.427$ m implies that one is sure about the first three digits (3.42) but somewhat uncertain about the last digit. Nonetheless, because one has *some* knowledge about the last digit, the length is said to have been measured to four significant figures. For any measured quantity there is always some uncertainty in the last digit given.

The rules for dealing with uncertainties in measured quantities is typically discussed in detail in the laboratory section of a course. We give an overview of the subject in the following problems.

Problem 1.29. Five lengths have been measured and recorded as follows:

$$L_1 = 3.427\ \text{m} \qquad L_2 = 3.5\ \text{m} \qquad L_3 = 0.333\ \text{m} \qquad L_4 = 12\ \text{m} \qquad L_5 = 32.000\ \text{m}$$

(*a*) Approximately what uncertainty is there in each measurement?

(*b*) What is the approximate percentage uncertainty in each measurement?

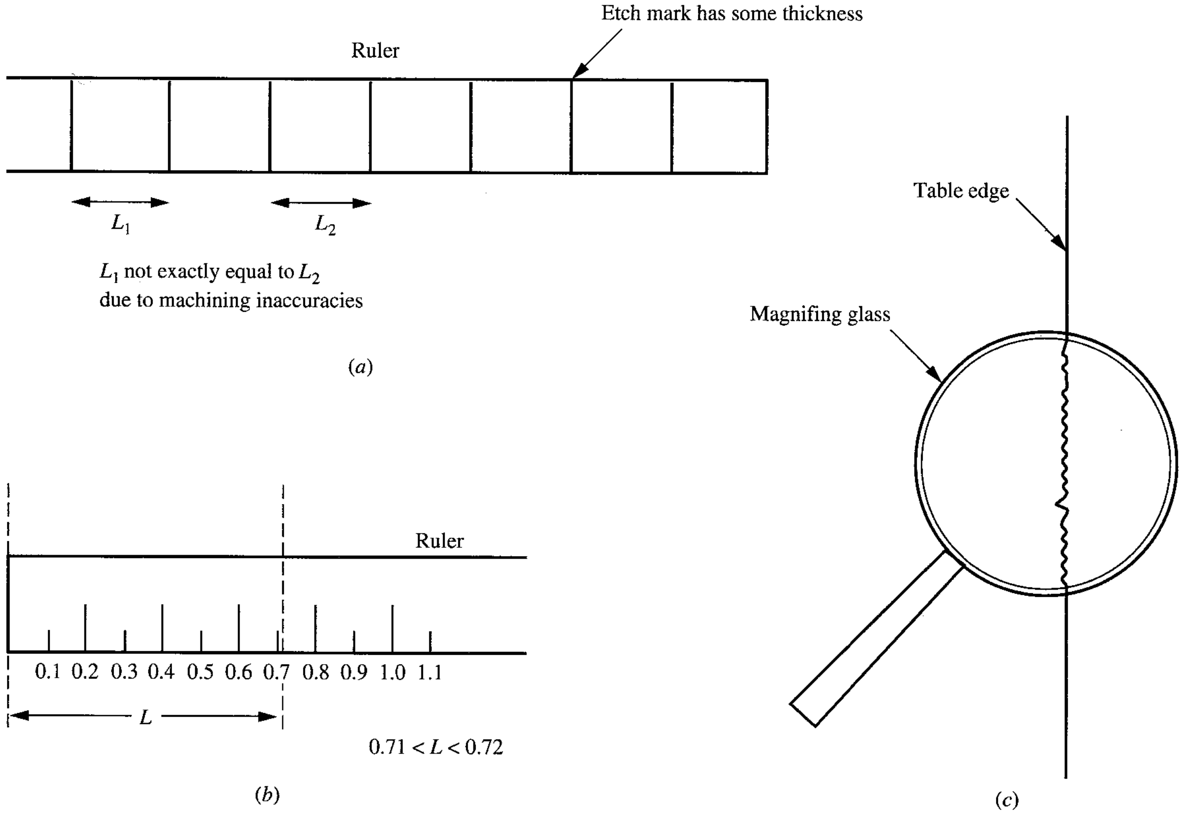


Fig. 1-15

Solution

(a) About one or two millimeters for L_1, L_3, and L_5; about one or two-tenths or a meter for L_2; and about one or two meters for L_4. Note the significance of placing the zeros after the decimal point in L_5. Although mathematically they don't change the value of L_5, they indicate the accuracy to which L_5 has been determined.

(b) For L_1 suppose that the uncertainty is two millimeters. Two out of 3427 corresponds to about six out of ten thousand, or multiplying by 100 to get percent, six hundredths of one percent. Because two millimeters is merely an estimate of the uncertainty, we can say that the percentage uncertainty in 3.427 m is several hundredths of one percent. Similarly for L_2 we have an uncertainty of, say two out of 35, which corresponds to several percent. For L_3 we have an uncertainty of about two out of 333, which corresponds to several tenths of one percent. For L_4 we have about two out of twelve uncertainty, which is about 20 percent. For L_5 we have about two parts in 32,000 uncertainty, which corresponds to several thousandths of one percent.

The number of significant figures thus provides a rough measure of percent uncertainty: two significant figures indicates several percent uncertainty, three significant figures indicates several tenths of a percent uncertainty; etc. The uncertainty itself indicates that the true value can be greater than or less than the recorded value by the amount of the uncertainty. The same holds for percent uncertainties.

Problem 1.30. For the lengths given in the previous problem:

(*a*) If one adds L_1 and L_3, how should one record the result?

(*b*) If one adds L_1 and L_2, how should one record the result?

Solution

(*a*) $L_1 + L_3 = 3.427\text{ m} + 0.333\text{ m} = 3.760\text{ m}$. Since each length is accurate to within a few millimeters, their sum is also. The last zero in the sum should be kept; otherwise an uncertainty of a few centimeters would be implied.

(*b*) $L_1 + L_2 = 3.427\text{ m} + 3.5\text{ m} = 6.9\text{ m}$. No chain is stronger than its weakest link, and here the uncertainty in L_2 dominates. It is already tenths of a meter, and the sum of the lengths can't have greater certainty than that.

Problem 1.31. For the lengths of Problem 1.29:

(*a*) If one subtracts L_3 from L_1, how should one record the result?

(*b*) Repeat for $L_2 - L_1$.

Solution

(*a*) $L_1 - L_3 = 3.427\text{ m} - 0.333\text{ m} = 3.094\text{ m}$. Since each length is accurate to a few millimeters, the difference is also accurate to a few millimeters, and hence the millimeter digit, 4, is kept.

(*b*) $L_2 - L_1 = 3.5\text{ m} - 3.427\text{ m} = 0.1\text{ m}$. As in Problem 1.30(*b*) we need to keep only one place to the right of the decimal point.

In this case it is interesting to examine the percentage uncertainty in the result. Since there is only one significant figure, the uncertainty is one or two parts out of one! Thus the uncertainty is greater than 100%, even though the percent uncertainty in L_2 is only a few percent. The result of 0.1 m may actually be as high as 0.3 m or as low as -0.1 m (meaning that L_2 might actually have been smaller than L_1). This drastic loss of accuracy occurs when one subtracts two measured quantities that are very close in value, producing a difference that is comparable to the uncertainties in the individual values.

Problem 1.32. Suppose that L_1 and L_2 of Problem 1.29 refer to the length and width of a rectangular tabletop. What is the area of the tabletop and how should it be recorded?

Solution

Area $= L_1 \cdot L_2 =$? The question is how many significant figures should be kept in the product. This can be answered by noting that the percent uncertainty in the product should be roughly the same as the larger of the percent uncertainties in the two factors. To understand why this is reasonable, let as assume that L_1 is known perfectly (not a real possibility). Then if L_2 were increased by 10% the area would be increased by 10% also (see Fig. 1-16). If, on the other hand, L_2 were reduced by, say, 20%, then the area would be reduced by 20% also. If L_1 were not exactly known but had an error of, say, several hundreths of a percent, as in our actual case, this would have a negligible effect on the percent uncertainty in the product. Thus, the percent uncertainty in the product is indeed approximately the same as the larger of the individual percent uncertainties.

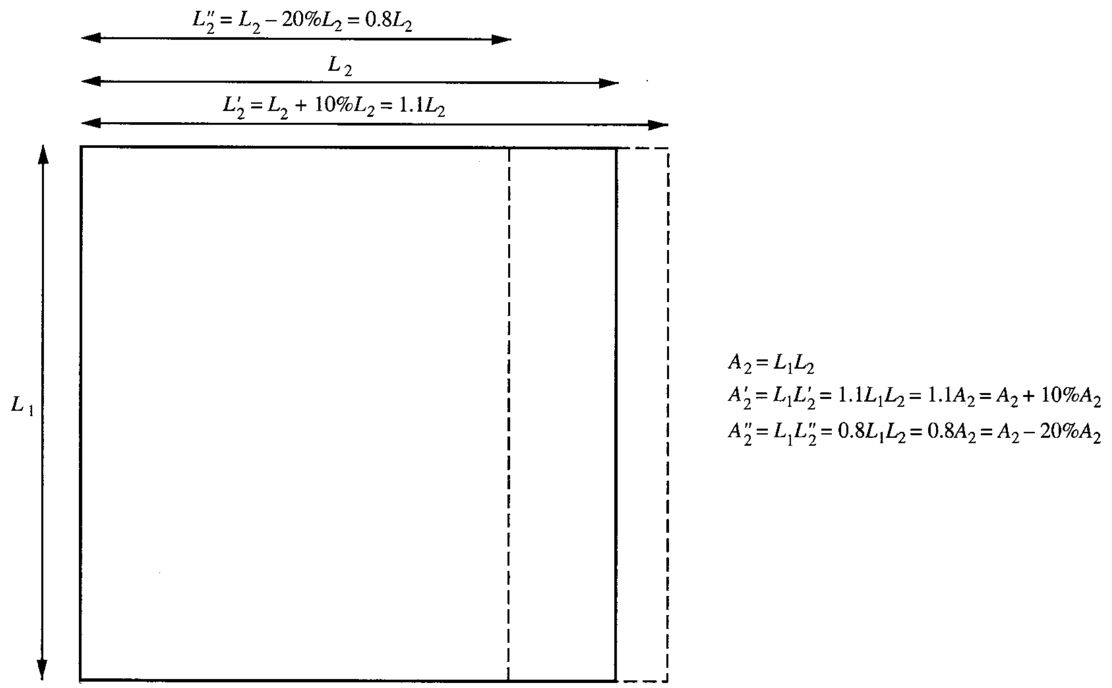


Fig. 1-16

For our case the larger percent uncertainty is in L_2, where it was shown in Problem 1.29 to be several percent. Then, following our analysis (after Problem 1.29) we should keep *at most* three significant figures in the product:

$$A = (3.427\ \text{m})\,(3.5\ \text{m}) = 12.0\ \text{m}^2$$

One could argue that we should keep only two significant figures in Problem 1.32 but this would imply a larger percentage uncertainty than is justified. While one can always determine in a given problem whether to do so or not, keeping one more significant figure in the product than found in the cruder factor is often a reasonable thing to do.

The same rule that holds for the number of significant figures in the product of two numbers holds for the quotient of two numbers as well. This is because, as in multiplication, the percent uncertainty in the quotient is roughly the same as the larger of the percent uncertainties in the quantities being divided.

Problem 1.33. A bicycle travels 634.73 ft in 42 s. What is the speed of the bicycle?

Solution

$$\text{Speed} = \frac{\text{distance}}{\text{time}} = \frac{634.73\ \text{ft}}{42\ \text{s}} = 15.1\ \text{ft/s}$$

Note. Often in a physics problem certain numbers appear that are to be presumed exact. Consider the circumference of a circle $C = 2\pi R$, where R is the measured radius. The 2 and the π are exact mathematical quantities, not measured quantities. Suppose, for example, R is given as $R = 2.16$ m. What is the circumference C? The number of significant figures in the answer will be the same as or at most one more than that of R (to reflect the *percentage* uncertainty in R). Since π is an infinite decimal, we keep only as many places as are necessary to not lessen the accuracy of the answer. In this case three-place accuracy suffices:

$$C = 2(3.14)\,(2.16\text{ m}) = 13.6$$

where we have rounded up after eliminating the last digit.

A practical note for students. There are times when an instructor will give a problem in which a physical quantity is presumed to be more accurate than the number of significant figures specified. For example, consider the following problem: "An automobile travels at a speed of 32.5 m/s. How far does it travel in 3 s?" This could be a trick question by the instructor to see if you remember how to deal with significant figures, but more likely the intention is for you to assume the time is given to at least the same accuracy as the speed. Always check with your instructor if you are not sure of the intention.

Scientific Notation

Sometimes there is a natural ambiguity as to the intended number of significant figures in a reported value. Suppose you are told the length of a field is 3200 m. The last two zeros may be significant figures, or they may merely show you where the decimal point is. This ambiguity can be avoided by specifying the length in *scientific notation.* In this notation every number is expressed with exactly one digit to the left of the decimal point, and then multiplied by the appropriate power of 10. For example, the number 356 is expressed as $3.56 \cdot 10^2$, and the length 0.0003246 cm is expressed as $3.246 \cdot 10^{-4}$ cm. The power of 10, called the **exponent**, can be positive or negative, and tells you how many digits to move the decimal point to the right or left, respectively. To add or subtract two numbers in scientific notation, one first has to convert them to numbers with the same exponent. For example,

$$3.56 \cdot 10^2 + 2.437 \cdot 10^3 = 0.356 \cdot 10^3 + 2.437 \cdot 10^3 = 2.793 \cdot 10^3$$

To multiply or divide two numbers is particularly easy since multiplying or dividing powers of 10 is accomplished by adding or subtracting the exponents, respectively.

Problem 1.34. Express in scientific notation the product of (*a*) 356 and 2000, (*b*) 356 and 0.0000200; find the quotient of (*c*) 356 divided by 2000, (*d*) 356 divided by 0.0000200.

Solution

(*a*) $(3.56 \cdot 10^2)\,(2.000 \cdot 10^3) = 7.12 \cdot 10^5$ (*c*) $\dfrac{3.56 \cdot 10^2}{2.000 \cdot 10^3} = 1.78 \cdot 10^{-1}$

(*b*) $(3.56 \cdot 10^2)\,(2.00 \cdot 10^{-5}) = 7.12 \cdot 10^{-3}$ (*d*) $\dfrac{3.56 \cdot 10^2}{2.00 \cdot 10^{-5}} = 1.78 \cdot 10^7$

Problems for Review and Mind Stretching

Problem 1.35. Variables y and x are related by the equation

$$6y + 3x = 12 \qquad (i)$$

(*a*) Write y as a function of x.

(*b*) Calculate the value of y at the following x values: $x = -2, -1, 0, 1, 2$.

(*c*) Find the slope and y intercept of the straight-line graph of the function.

Solution

(*a*) Isolate y by first subtracting $3x$ from each side of (*i*): $6y = -3x + 12$. Next, divide both sides by 6 to get

$$y = f(x) = -\tfrac{1}{2}x + 2 \qquad (ii)$$

which is the desired result.

(*b*) Substitute the values into (*ii*) to get

$$y = f(-2) = -\tfrac{1}{2}(-2) + 2 = 3 \qquad y = f(1) = -\tfrac{1}{2}(1) + 2 = 1.5$$
$$y = f(-1) = -\tfrac{1}{2}(-1) + 2 = 2.5 \qquad y = f(2) = -\tfrac{1}{2}(2) + 2 = 1$$
$$y = f(0) = -\tfrac{1}{2}(0) + 2 = 2$$

(*c*) $y = mx + b$, where m is the slope and b is the y-intercept. For our case, (*ii*) above, we have slope $= -\frac{1}{2}$, y intercept $= 2$.

Problem 1.36. Find the equation of the straight line shown in Fig. 1-17.

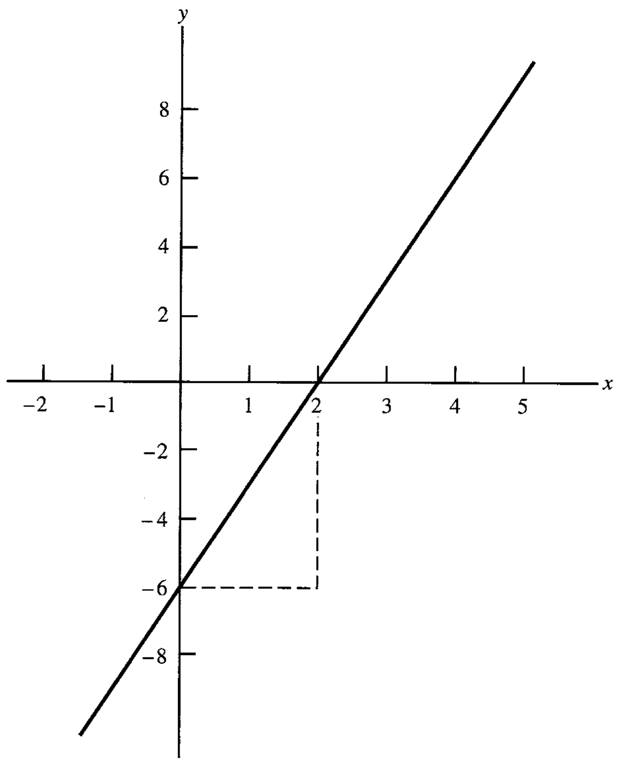


Fig. 1-17

Solution

For a general straight line, $y = Ax + B$. From the graph we see that when $x = 0$, $y = -6$, so the intercept is $B = -6$. The slope A can be obtained by considering the two points where the line crosses the axes and the right triangle formed by the dashed lines in the figure.

$$\text{Slope} = \frac{\text{vertical rise}}{\text{horizontal shift}} = \frac{6}{2} = 3$$

so $A = 3$. Then our equation is $y = 3x - 6$.

Problem 1.37. Given the function $y = f(x) = 4x^2 + 2$.

(*a*) Show that the graph of the function is symmetric about the y axis, and find the smallest value of y.

(*b*) Find the inverse function $x = f^{-1}(y)$, and sketch what it looks like on a graph.

Solution

(*a*) Since y takes on the same values at the points x and $-x$, the curve will be symmetric about a vertical line through $x = 0$, or in other words, about the y axis. y will be smallest when the term $4x^2$ is smallest, and that occurs at $x = 0$. Thus, $y_{\text{min}} = 2$.

(*b*) $$y = 4x^2 + 2 \Rightarrow 4x^2 = y - 2 \Rightarrow x^2 = 0.25y - 0.5 \qquad \text{or} \qquad x = \pm\sqrt{0.25y - 0.5}$$

So we have two branches of the inverse function:

$$x = f_1{}^{-1}(y) = \sqrt{-0.25y - 0.5} \qquad \text{and} \qquad x = f_2{}^{-1}(y) = -\sqrt{0.25y - 0.5}$$

defined for all values of $y \geq 2$. A rough graph of the inverse function is shown in Fig. 1-18.

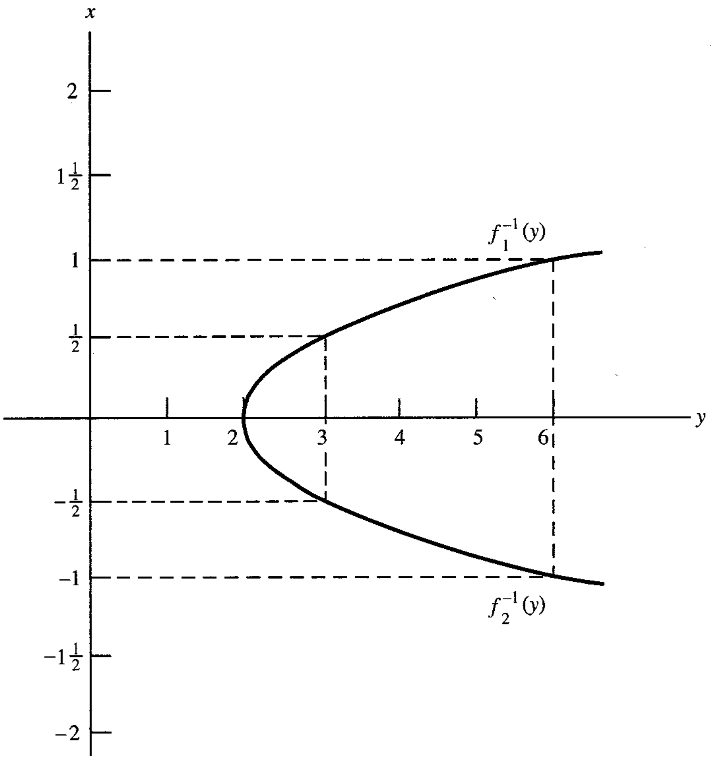


Fig. 1-18

Problem 1.38.

(*a*) Find the sine, cosine, and tangent of the angle θ shown in Fig. 1-19.

(*b*) Repeat for angle ϕ.

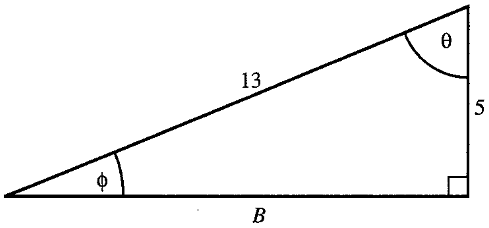


Fig. 1-19

Solution

(*a*) Since $\cos\theta = a/h$, we have $\cos\theta = \frac{5}{13} = 0.385$. To obtain $\sin\theta$ and $\tan\theta$ we need the side B opposite θ. Using the pythagorean theorem

$$5^2 + B^2 = 13^2 \quad \text{or} \quad B^2 = 13^2 - 5^2 = 169 - 25 = 144 \Rightarrow B = 12$$

Then $\sin\theta = \frac{12}{13} = 0.923$; $\tan\theta = \frac{12}{5} = 2.40$.

(*b*) We could repeat the calculations of part (*a*) for the case of ϕ. Instead we note that since $\phi = 90° - \theta$ (complementary angles), we have

$$\cos\phi = \sin\theta = 0.923 \qquad \sin\phi = \cos\theta = 0.385 \qquad \tan\phi = \frac{1}{\tan\theta} = 0.417$$

Problem 1.39. Determine the angles θ and ϕ in Fig. 1-19.

Solution

Using the results of Prob. 1.38, we have $\theta = \sin^{-1}(0.923)$. Utilizing the inverse sine function on a calculator (or looking up θ in a trigonometric table), we get $\theta = 67.4°$. Then, $\phi = 90° - \theta = 22.6°$. As a check we calculate $\tan 22.6° = 0.416$, which agrees with Problem 1.38(*b*) to within rounding errors.

Problem 1.40. Solve the simultaneous equations

$$\text{(i)}\quad y = 5t - 7 \qquad \text{(ii)}\quad y = t^2 - 1$$

Solution

Since both right sides equal y, they equal each other: $t^2 - 1 = 5t - 7$. Bringing all terms to the left side of the equation, we get

$$\text{(iii)}\quad t^2 - 5t + 6 = 0$$

This can be factored, yielding $(t - 3)(t - 2) = 0$, and the two solutions for t are $t_1 = 3$ and $t_2 = 2$. Then, from either (i) or (ii) we get the corresponding values of y:

$$y_1 = 8 \quad \text{and} \quad y_2 = 3$$

Problem 1.41. Suppose in Eq. (*1.8*) (preceding Problem 1.27) we are given $a = 6.25\ \text{m/s}^2$ and told that $x = 122$ m when $t = 3.15$ s. Find v_0.

Solution

$$\begin{aligned} x &= v_0 t + \tfrac{1}{2}at^2 \\ 122\ \text{m} &= v_0(3.15\ \text{s}) + \tfrac{1}{2}(6.25\ \text{m/s}^2)\,(3.15\ \text{s})^2 \\ 122\ \text{m} &= v_0(3.15\ \text{s}) + 31.0\ \text{m} \\ 91\ \text{m} &= (3.15\ \text{s})v_0 \\ v_0 &= 28.9\ \text{m/s} \end{aligned}$$

Problem 1.42. Two measured lengths are recorded as $l_1 = 23.2$ m and $l_2 = 21.749$ m.

(*a*) How big an uncertainty would you roughly estimate there is in the value of l_1? Repeat for l_2.

(*b*) How big, roughly, are the percent uncertainties in l_1 and l_2?

(*c*) If the two lengths are to be added, how would one record their sum L?

(*d*) What is the uncertainty in the recorded sum?

Solution

(*a*) Assume an uncertainty between 1 and 2 in the last significant digit of each number. Then for l_1 the uncertainty is 0.1 to 0.2 m. For l_2 it is 0.001 to 0.002 m.

(*b*) For definiteness we use 0.2 m as the uncertainty in l_1. The percent uncertainty is then (0.2/23.2) (100) = 0.86%. Hence, there is about a 1% uncertainty in l_1. For l_2 we get about (0.002/22) (100) = 0.009%. Hence, l_2 has about a 0.01% uncertainty.

(*c*) $L = l_1 + l_2 = 23.2\ \text{m} + 21.749\ \text{m} = 44.9\ \text{m}$.

(*d*) The uncertainty of the sum should be roughly that of the less precisely known length. Therefore the uncertainty in L is about 0.1 to 0.2 m. (Indeed, this is precisely why we keep the sum to three significant figures.)

Problem 1.43. Assume l_1 and l_2 of Problem 1.42 are the length and width of a swimming pool.

(*a*) How would you record the area A?

(*b*) What is the percent uncertainty in the area?

(*c*) What is the uncertainty in the area?

Solution

(*a*) $\text{A} = l_1 l_2 = (23.2\ \text{m})\ (21.749\ \text{m}) = 505\ \text{m}$.

(*b*) The percent uncertainty in a product of two factors is about the same as the larger percent uncertainty of the two factors. In our case this is l_1, and from Problem 1.42 the percent uncertainty is about 1%.

(*c*) Since the percent uncertainty in the area is about 1%, and the area is about 500 m^2, the uncertainty in the area is about 1% of 500 m^2 or about 5 m^2.

Problem 1.44. Express in scientific notation (*a*) the lengths $l_1 = 436.37$ m and $l_2 = 0.00169$ m, (*b*) the product of l_1 and l_2, and (*c*) the ratio of l_1 to l_2.

Solution

(*a*) $$l_1 = 4.3637 \cdot 10^2 \text{ m} \qquad l_2 = 1.69 \cdot 10^{-3} \text{ m}$$

(*b*) $$l_1 l_2 = (4.3637 \cdot 10^2 \text{ m})\,(1.69 \cdot 10^{-3} \text{ m}) = 7 \cdot 37 \cdot 10^{-1} \text{ m}^2$$

(*c*) $$\frac{l_1}{l_2} = \frac{4.3637 \cdot 10^2 \text{ m}}{1.69 \cdot 10^{-3} \text{ m}} = 2.58 \cdot 10^5$$

Note that in parts (*b*) and (*c*) we keep three significant figures, and that in part (*c*) the answer has no units (is dimensionless) since it is the ratio of two lengths.

Supplementary Problems

Problem 1.45. Given the function $y = f(x) = 10x - 24$, find (*a*) its slope and y intercept; (*b*) the inverse function $x = f^{-1}(y)$; (*c*) the slope and x intercept of the inverse function.

Ans. (*a*) 10 and -24; (*b*) $x = 0.1y + 2.4$; (*c*) 0.1 and 2.4

Problem 1.46. Given the equation $12x + 16y = 24$, find y as a function of x, and x as a function of y. Are these two functions inverses of each other?

Ans. $y = -0.75x + 1.5$; $x = -(\frac{4}{3})y + 2$; yes

Problem 1.47. Find the inverse of the function found in (ii) of Problem 1.35.

Ans. $x = -2y + 4$

Problem 1.48. Find both branches of the inverse function of $z = t^2 - 4$. For what values of z is this defined?

Ans. $\pm\sqrt{z+4}$; $z \geq -4$

Problem 1.49. In the triangle shown in Fig. 1-20, find sides A and B.

Ans. 26.0; 15.0

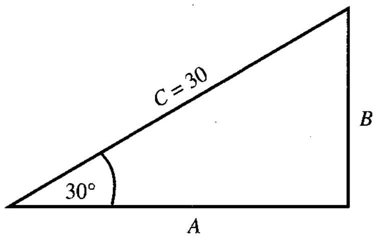


Fig. 1-20

√ **Problem 1.50.** A student wants to determine the height of a flagpole in the schoolyard. She paces off 100 ft from the base of the flagpole and then measures the angle between the gound and a line of sight to the top of the flagpole to be 30°.

(*a*) What is the height of the flagpole?

(*b*) If, instead, the flagpole had been 85 ft tall, what angle would she have gotten?

Ans. (*a*) 57.7 ft.; (*b*) 40.4°

Problem 1.51. If $\sin\theta = 0.90$ and θ is acute, find $\cos\theta$ and $\tan\theta$ without first finding θ.

Ans. $\cos\theta = 0.44$; $\tan\theta = 2.05$

Problem 1.52. Convert each of the following to the sine of an acute angle: (i) cos 55°, (ii) sin 135°, (iii) sin 206°, (iv) sin 340°, (v) sin(−40°).

Ans. (i) sin 35°, (ii) sin 45°, (iii) −sin 26°, (iv) −sin 20°, (v) −sin 40°

Problem 1.53. Convert each of the following to the cosine of an acute angle: (i) sin 15°, (ii) cos 128°, (iii) cos 199°, (iv) cos 295°, (v) cos(−130°).

Ans. (i) cos 75°, (ii) −cos 52°, (iii) −cos 19°, (iv) cos 65°, (v) −cos 50°

Problem 1.54. Convert each of the following to the tangent of an acute angle: (i) tan 170°, (ii) tan 250°, (iii) tan 310°, (iv) tan(−25°), (v) cot 22°.

Ans. (i) −tan 10°, (ii) tan 70°, (iii) −tan 50°, (iv) −tan 25°, (v) tan 68°

Problem 1.55. Find the solution of the pair of equations $5y + 8x = 1$ and $4y - 2x = 5$.

Ans. $x = -\frac{1}{2}$, $y = 1$

√ **Problem 1.56.** Solve the simultaneous equations $6x + y = 2$ and $2x + 5y = 3$.

Ans. $x = \frac{1}{4}$; $y = \frac{1}{2}$

Problem 1.57. Find the solutions of the pair of equations $y - 3x = 1$ and $y = x^2 - 9$.

Ans. $(x = -2, y = -5)$; $(x = 5, y = 16)$

Problem 1.58. Find all the solutions to the pair of equations $y = 3x^2 - 4$ and $y = 2x^2 + 2x + 4$.

Ans. $(x = -2, y = 8)$; $(x = 4, y = 44)$

√ **Problem 1.59.** A snail moves at a speed of 80 ft/h. How many meters will it travel in 100 s?

Ans. 0.677 m

Problem 1.60. A rectangular block has dimensions $L = 3.24$ ft, $W = 39.2$ cm, $H = 1.62$ m. Find the volume V in m^3.

Ans. 0.627 m^3

Problem 1.61.

(*a*) If the length l_2 in Problem 1.42 were to be subtracted from the length l_1, how should the resulting length L' be recorded?

(*b*) What, roughly, is the uncertainty in L'?

(*c*) Estimate the percent uncertainty in the length L'.

Ans. (*a*) 1.5 m; (*b*) 0.1 to 0.4 m; (*c*) 10 to 30%

Problem 1.62. Give a rough estimate of the uncertainty and a percent uncertainty in each of the following measured quantities: (*a*) 1.8307 m; (*b*) 321 s; (*c*) 12 ft; (*d*) 0.000223 m.

Ans. (*a*) 0.0002 m, 0.01%; (*b*) 2 s, 1%; (*c*) 2 ft, 20%; (*d*) 0.000002 m, 1%

Problem 1.63. Assuming an uncertainty of 2 in the last digit of each measurement of Problem 1.62, find its range of possible values.

Ans. (*a*) 1.8305 to 1.8309 m; (*b*) 319 to 323 s; (*c*) 10 to 14 ft; (*d*) 0.000221 to 0.000225 m

Problem 1.64. The length of a rug is measured to be 3.1944 m. The width is measured to be 6.22 ft.

(*a*) Find the area in m^2.

(*b*) Find the approximate percent uncertainty in the area.

(*c*) Give a rough estimate of the actual uncertainty in the area.

Ans. (*a*) 6.06 m^2; (*b*) about 0.5%; (*c*) about 0.03 m^2

Problem 1.65.

(*a*) Find the perimeter of the rug in Problem 1.64 in meters.

(*b*) Is the percent uncertainty in the perimeter somewhat smaller or somewhat larger than the percent uncertainty in the width?

Ans. (*a*) 10.18 m; (*b*) somewhat smaller

Problem 1.66. Referring to Problem 1.50(*a*), assume that the uncertainty in the measured angle is 2°. If the 100-ft distance is known with great accuracy, what is the uncertainty in the calculated height of the flagpole?

Ans. 5 ft

Problem 1.67. Given the formula $z = xy/t^2$, let $x = 132$, $y = 0.00736$, and $t = 0.0955$.

(*a*) Write x, y, and t, in scientific notation.

(*b*) Find z in scientific notation.

Ans. (*a*) $1.32 \cdot 10^2$; $7.36 \cdot 10^{-3}$; $9.55 \cdot 10^{-2}$; (*b*) $1.065 \cdot 10^2$

Problem 1.68. Given the formula $x = v_0 t + \frac{1}{2}at^2$, suppose that $v_0 = 32.666$ m/s, $a = .00177$ m/s^2, and $t = 322$ s. Convert all quantities to scientific notation and determine the value of x.

Ans. $3.2666 \cdot 10^1$ m/s; $1.77 \cdot 10^{-3}$ m/s^2; $3.22 \cdot 10^2$ s; $1.06 \cdot 10^4$ m