BOOK II

THE CIRCLE

159. Circle. A closed curve lying in a plane, and such that all of its points are equally distant from a fixed point in the plane, is called a *circle.*

160. Circle as a Locus. It follows that *the locus of a point in a plane at a given distance from a fixed point is a circle.*

161. Radius. A straight line from the center to the circle is called a *radius.*

162. Equal Radii. It follows that *all radii of the same circle or of equal circles are equal,* and that *all circles of equal radii are equal.*

163. Diameter. A straight line through the center, terminated at each end by the circle, is called a *diameter.*

Since a diameter equals two radii, it follows that *all diameters of the same circle or of equal circles are equal.*

164. Arc. Any portion of a circle is called an *arc.*

An arc that is half of a circle is called a *semicircle.*

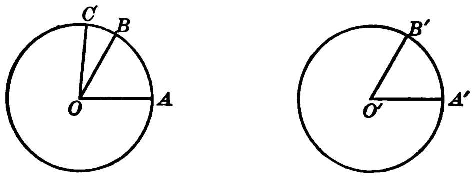
An arc less than a semicircle is called a *minor arc,* and an arc greater than a semicircle is called a *major arc.* The word *arc* taken alone is generally understood to mean a minor arc.

165. Central Angle. If the vertex of an angle is at the center of a circle and the sides are radii of the circle, the angle is called a *central angle.*

An angle is said to *intercept* any arc cut off by its sides, and the arc is said to *subtend* the angle.

Proposition I. Theorem

166. *In the same circle or in equal circles equal central angles intercept equal arcs; and of two unequal central angles the greater intercepts the greater arc.*



Given two equal circles with centers O and O', with angles AOB and $A'O'B'$ equal, and with angle AOC greater than angle $A'O'B'$.

To prove that 1. *arc* $AB =$ *arc* $A'B'$;

2. *arc* $AC >$ *arc* $A'B'$.

Proof. 1. Place the circle with center O on the circle with center O' so that $\angle AOB$ shall coincide with its equal, $\angle A'O'B'$. In the case of the same circle, swing one angle about O until it coincides with its equal angle. Post. 5

Then A falls on A', and B on B'. § 162

(Radii of equal circles are equal.)

$\therefore$ arc AB coincides with arc $A'B'$. § 159

(Every point of each is equally distant from the center.)

Proof. 2. Since $\angle AOC$ is greater than $\angle A'O'B'$, Given

and $\angle AOB = \angle A'O'B'$, Given

therefore $\angle AOC$ is greater than $\angle AOB$. Ax. 9

Therefore OC lies outside $\angle AOB$.

$\therefore$ arc $AC >$ arc AB. Ax. 11

But arc $AB =$ arc $A'B'$.

$\therefore$ arc $AC >$ arc $A'B'$, by Ax. 9. Q.E.D.

PROPOSITION II. THEOREM

167. *In the same circle or in equal circles equal arcs subtend equal central angles; and of two unequal arcs the greater subtends the greater central angle.*

Given two equal circles with centers O and O', with arcs AB and $A'B'$ equal, and with arc AC greater than arc $A'B'$.

To prove that 1. $\angle AOB = \angle A'O'B'$;

2. $\angle AOC > \angle A'O'B'$.

Proof. 1. Using the figure of Prop. I, place the circle with center O on the circle with center O' so that OA shall fall on its equal $O'A'$, and the arc AB on its equal $A'B'$. Post. 5

Then OB coincides with $O'B'$. Post. 1

$\therefore \angle AOB = \angle A'O'B'$. § 23

Proof. 2. Since arc $AC >$ arc $A'B'$, it is greater than arc AB, the equal of arc $A'B'$, and OB lies within the $\angle AOC$. Ax. 9

$\therefore \angle AOC > \angle AOB$. Ax. 11

$\therefore \angle AOC > \angle A'O'B'$, by Ax. 9. Q.E.D.

This proposition is the converse of Prop. I.

168. Law of Converse Theorems. Of four magnitudes, a, b, x, y, if

(1) $a > b$ when $x > y$,

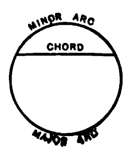
(2) $a = b$ when $x = y$,

and (3) $a < b$ when $x < y$,

then *the converses of these three statements are always true.*

For when $a > b$ it is impossible that $x = y$, for then a would equal b by (2); or that $x < y$, for then a would be less than b by (3). Hence $x > y$ when $a > b$. In the same way, $x = y$ when $a = b$, and $x < y$ when $a < b$.

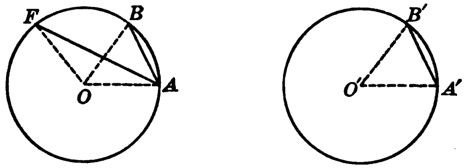
169. Chord. A straight line that has its extremities on a circle is called a *chord*.



A chord is said to *subtend* the arcs that it cuts from a circle. Unless the contrary is stated, the chord is taken as subtending the minor arc.

PROPOSITION III. THEOREM

170. *In the same circle or in equal circles, if two arcs are equal, they are subtended by equal chords; and if two arcs are unequal, the greater is subtended by the greater chord.*



Given two equal circles with centers O and O', with arcs AB and $A'B'$ equal, and with arc AF greater than arc $A'B'$.

To prove that 1. *chord* $AB =$ *chord* $A'B'$;

2. *chord* $AF >$ *chord* $A'B'$.

Proof. 1. Draw the radii OA, OB, OF, $O'A'$, $O'B'$.

Since $OA = O'A'$, and $OB = O'B'$, § 162

and $\angle AOB = \angle A'O'B'$, § 167

(*In equal* ⊛ *equal arcs subtend equal central* ∡.)

$\therefore \triangle OAB$ is congruent to $\triangle O'A'B'$, § 68

and chord $AB =$ chord $A'B'$. § 67

Proof. 2. In the ∆ OAF and $O'A'B'$,

$OA = O'A'$, and $OF = O'B'$, § 162

but $\angle AOF$ is greater than $\angle A'O'B'$. § 167

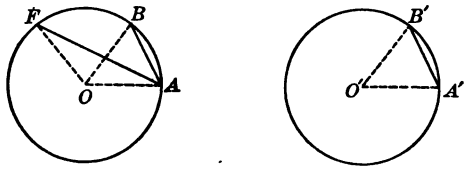
(*In equal* ⊛, *of two unequal arcs the greater subtends the greater central* ∠.)

$\therefore$ chord $AF >$ chord $A'B'$, by § 115. Q.E.D.

171. COROLLARY. *In the same circle or in equal circles, the greater of two unequal major arcs is subtended by the less chord.*

Proposition IV. Theorem

172. *In the same circle or in equal circles, if two chords are equal, they subtend equal arcs; and if two chords are unequal, the greater subtends the greater arc.*



Given two equal circles with centers O and O', with chords AB and $A'B'$ equal, and with chord AF greater than chord $A'B'$.

To prove that 1. *arc* $AB =$ *arc* $A'B'$;

2. *arc* $AF >$ *arc* $A'B'$.

Proof. 1. Draw the radii OA, OB, OF, $O'A'$, $O'B'$.

Since	$OA = O'A'$, and $OB = O'B'$,	§ 162
and	chord $AB =$ chord $A'B'$,	Given
	$\therefore \triangle OAB$ is congruent to $\triangle O'A'B'$,	§ 80
and	$\angle AOB = \angle A'O'B'$.	§ 67
	$\therefore$ arc $AB =$ arc $A'B'$.	§ 166

Proof. 2. In the $\triangle$s OAF and $O'A'B'$,

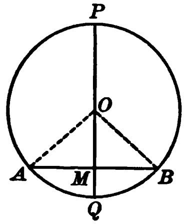
	$OA = O'A'$, and $OF = O'B'$,	§ 162
but	chord $AF >$ chord $A'B'$.	Given
	$\therefore \angle AOF > \angle A'O'B'$.	§ 116
	$\therefore$ arc $AF >$ arc $A'B'$, by § 166.	Q.E.D.

This proposition is the converse of Prop. III.

173. Corollary. *In the same circle or in equal circles the greater of two unequal chords subtends the less major arc.*

Proposition V. Theorem

174. *A line through the center of a circle perpendicular to a chord bisects the chord and the arcs subtended by it.*



Given the line *PQ* through the center *O* of the circle *AQBP*, perpendicular to the chord *AB* at *M*.

To prove that $AM = BM$, arc AQ = arc BQ, and arc AP = arc BP.

Proof. Draw the radii OA and OB.

Then since	$OM = OM$,	Iden.
and	$OA = OB$,	§ 162
	$\therefore$ rt. $\triangle AMO$ is congruent to rt. $\triangle BMO$.	§ 89
	$\therefore AM = BM$, and $\angle AOQ = \angle QOB$.	§ 67
Likewise	$\angle POA = \angle BOP$.	§ 58

$\therefore$ arc AQ = arc BQ, and arc AP = arc BP, by § 166. Q.E.D.

175. Corollary 1. *A diameter bisects the circle.*

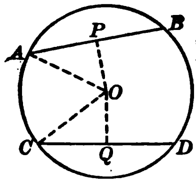
176. Corollary 2. *A line through the center that bisects a chord, not a diameter, is perpendicular to the chord.*

177. Corollary 3. *The perpendicular bisector of a chord passes through the center of the circle and bisects the arcs subtended by the chord.*

How many bisectors of the chord are possible? How many $\perp$ bisectors? Therefore with what line must this coincide (§ 174)?

PROPOSITION VI. THEOREM

178. *In the same circle or in equal circles equal chords are equidistant from the center, and chords equidistant from the center are equal.*



Given AB and CD, equal chords of the circle $ACDB$.

To prove that AB and CD are equidistant from the center O.

Proof. Draw $OP \perp$ to AB, and $OQ \perp$ to CD.

Draw the radii OA and OC.

OP bisects AB, and OQ bisects CD. § 174

Then since $AP = CQ$, Ax. 4

and $OA = OC$, § 162

$\therefore$ rt. $\triangle OPA$ is congruent to rt. $\triangle OQC$. § 89

$\therefore OP = OQ$. § 67

$\therefore AB$ and CD are equidistant from O, by § 88. Q.E.D.

Given OP and OQ, equal perpendiculars from the center O to the chords AB and CD.

To prove that $AB = CD$.

Proof. Since $OA = OC$, § 162

and $OP = OQ$, Given

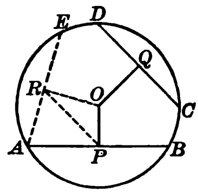
$\therefore$ rt. $\triangle OPA$ is congruent to rt. $\triangle OQC$. § 89

$\therefore AP = CQ$. § 67

$\therefore AB = CD$, by Ax. 3. Q.E.D.

PROPOSITION VII. THEOREM

179. *In the same circle or in equal circles, if two chords are unequal, they are unequally distant from the center, and the greater chord is at the less distance.*



Given a circle with center *O*, two unequal chords *AB* and *CD*, *AB* being the greater, and *OP* perpendicular to *AB*, and *OQ* perpendicular to *CD*.

To prove that $OP < OQ$.

Proof. Suppose AE drawn equal to CD, and $OR \perp$ to AE.

Draw PR.

OP bisects AB, and OR bisects AE. § 174

(*A line through the center of a circle* $\perp$ *to a chord bisects the chord.*)

But $AB > CD$. Given

$\therefore AB > AE$, the equal of CD. Ax. 9

$\therefore AP > AR$. Ax. 6

$\therefore \angle ARP > \angle RPA$. § 113

(*If two sides of a* $\triangle$ *are unequal, the* $\angle$s *opposite these sides are unequal, and the* $\angle$ *opposite the greater side is the greater.*)

$\therefore \angle PRO$, the complement of $\angle ARP$, is less than $\angle OPR$, the complement of $\angle RPA$. § 59

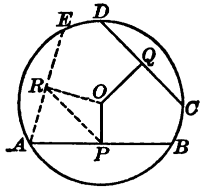
$\therefore OP < OR$. § 114

But $OR = OQ$. § 178

$\therefore OP < OQ$, by Ax. 9. Q.E.D.

Proposition VIII. Theorem

180. *In the same circle or in equal circles, if two chords are unequally distant from the center, they are unequal, and the chord at the less distance is the greater.*



Given a circle with center O, two chords AB and CD unequally distant from O, and OP, the perpendicular to AB, less than OQ, the perpendicular to CD.

To prove that $AB > CD$.

Proof. Suppose AE drawn equal to CD, and $OR \perp$ to AE.

Now	$OP < OQ$,	Given
and	$OR = OQ$.	§ 178
	$\therefore OP < OR$.	Ax. 9
Drawing PR,	$\angle PRO < \angle OPR$.	§ 113

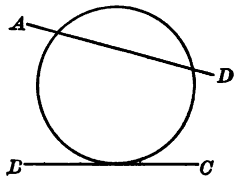
$\therefore \angle ARP$, the complement of $\angle PRO$, is greater than $\angle RPA$, the complement of $\angle OPR$. § 59

	$\therefore AP > AR$.	§ 114
But	$AP = \frac{1}{2} AB$, and $AR = \frac{1}{2} AE$.	§ 174
	$\therefore AB > AE$.	Ax. 6
But	$CD = AE$.	Hyp.
	$\therefore AB > CD$, by Ax. 9.	Q.E.D.

This proposition is the converse of Prop. VII.

181. Corollary. *A diameter of a circle is greater than any other chord.*

182. Secant. A straight line that intersects a circle is called a *secant.* In this figure AD is a secant.



Since only two equal obliques can be drawn to a line from an external point (§ 85), and since the two equal angles which radii make (§ 74) with any secant where it cuts the circle cannot be right angles (§ 109), they must be oblique; and hence it follows that *a secant can intersect the circle in only two points.*

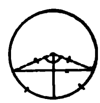
183. Tangent. A straight line of unlimited length that has one point, and only one, in common with a circle is called a *tangent* to the circle.

In this case the circle is said to be tangent to the line. Thus in the figure, BC is tangent to the circle, and the circle is tangent to BC.

The common point is called the *point of contact* or *point of tangency.*

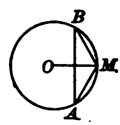
By the tangent from an external point to a circle is meant the line-segment from the external point to the point of contact.

EXERCISE 27

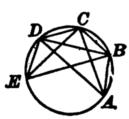


1. A radius that bisects an arc bisects its subtending chord and is perpendicular to it.

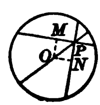
2. On a circle the point P is equidistant from two radii OA and OB. Prove that P bisects the arc AB.



3. In this circle the chords AM and MB are equal. Prove that M bisects the arc AB and that the radius OM bisects the chord AB.



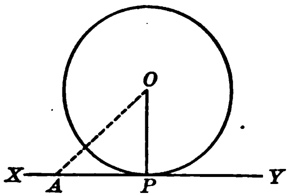
4. On a circle are five points, A, B, C, D, E, so placed that AB, BC, CD, DE are equal chords. Prove that AC, BD, CE are equal chords, and that AD and BE are also equal chords.



5. If two chords intersect and make equal angles with the diameter through their point of intersection, these chords are equal.

Proposition IX. Theorem

184. *A line perpendicular to a radius at its extremity on the circle is tangent to the circle.*



Given a circle, with XY perpendicular to the radius OP at P.

To prove that XY is tangent to the circle.

Proof. From O draw any other line to XY, as OA.

Then $OA > OP$. § 86

$\therefore$ the point A is outside the circle. § 160

Hence every point, except P, of the line XY is outside the circle.

Therefore XY is tangent to the circle at P, by § 183. Q.E.D.

185. Corollary 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

For OP is the shortest line from O to XY, and is therefore $\perp$ to XY (§ 86); that is, XY is $\perp$ to OP.

186. Corollary 2. *A perpendicular to a tangent at the point of contact passes through the center of the circle.*

For a radius is $\perp$ to a tangent at the point of contact, and therefore a $\perp$ erected at the point of contact coincides with this radius and passes through the center of the circle.

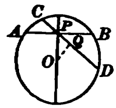
187. Corollary 3. *A perpendicular from the center of a circle to a tangent passes through the point of contact.*

What does § 86 say about this perpendicular?

188. Concentric Circles. Two circles that have the same center are said to be concentric.

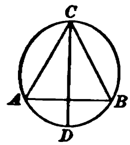
EXERCISE 28

1. The shortest chord that can be drawn through a given point within a circle is that which is perpendicular to the diameter through the point.



Show that any other chord, CD, through P, is nearer O than is AB.

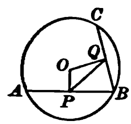
2. The diameter CD bisects the arc AB. Prove that $\angle CBA = \angle BAC$.



What kind of a triangle is $\triangle ABC$?

3. Tangents at the extremities of a diameter are parallel.

4. The arc AB is greater than the arc BC. OP and OQ are perpendiculars from the center to AB and BC respectively. Prove that $\angle QPO$ is greater than $\angle OQP$.

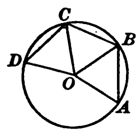


5. What is the locus of the center of a circle tangent to the line XY at the point P? Prove it.

What two conditions must be shown to be fulfilled?

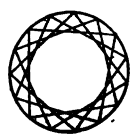
6. What is the locus of the mid-points of a number of parallel chords of a circle? Prove it.

7. Three equal chords, AB, BC, CD, are placed end to end, and the radii OA, OB, OC, OD are drawn. Prove that $\angle AOC = \angle BOD$.



8. All equal chords of a circle are tangent to a concentric circle.

9. If a number of equal chords are drawn in this circle, the figure gives the impression of a second circle inside the first and concentric with it. Explain the reason.



PROPOSITION X. THEOREM

189. *Two parallel lines intercept equal arcs on a circle.*

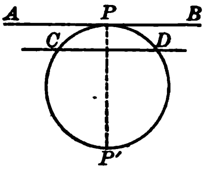


FIG. 1

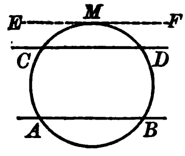


FIG. 2

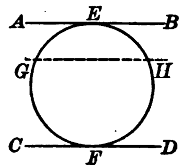


FIG. 3

CASE 1. *When the parallels are a tangent and a secant* (Fig. 1).

Given *AB*, a tangent at *P*, parallel to *CD*, a secant.

To prove that arc $CP =$ arc DP.

Proof. Suppose PP' drawn $\perp$ to AB at P.

Then PP' is a diameter of the circle. § 186

And PP' is also $\perp$ to CD. § 97

$\therefore$ arc $CP =$ arc DP. § 174

CASE 2. *When the parallels are both secants* (Fig. 2).

Given *AB* and *CD*, parallel secants.

To prove that arc $AC =$ arc BD.

Proof. Suppose $EF \parallel$ to CD and tangent to the circle at M.

Then arc $AM =$ arc BM, and arc $CM =$ arc DM. Case 1

$\therefore$ arc $AC =$ arc BD. Ax. 2

CASE 3. *When the parallels are both tangents* (Fig. 3).

Given *AB*, a tangent at *E*, parallel to *CD*, a tangent at *F*.

To prove that arc $FGE =$ arc FHE.

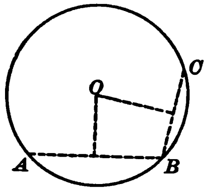
Proof. Suppose a secant GH drawn $\parallel$ to AB.

Then arc $GE =$ arc HE, and arc $FG =$ arc FH. Case 1

$\therefore$ arc $FGE =$ arc FHE, by Ax. 1. Q.E.D.

PROPOSITION XI. THEOREM

190. *Through three points not in a straight line one circle, and only one, can be drawn.*



Given A, B, C, three points not in a straight line.

To prove that one circle, and only one, can be drawn through A, B, and C.

Proof. Draw AB and BC.

At the mid-points of AB and BC suppose ⊥s erected.

These ⊥s will intersect at some point O, since AB and BC are neither parallel nor in the same straight line.

The point O is in the perpendicular bisector of AB, and is therefore equidistant from A and B; the point O is also in the perpendicular bisector of BC, and is therefore equidistant from B and C. § 150

Therefore O is equidistant from A, B, and C.

Therefore a circle described about O as a center, with a radius OA, will pass through the three given points. § 160

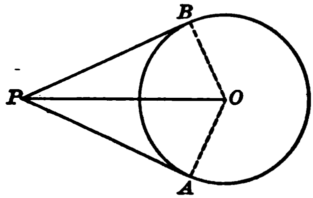
The center of any circle that passes through the three points must be in both of these perpendicular bisectors, and hence at their intersection. As two straight lines can intersect in only one point (§ 55), O is the only point that can be the center of a circle through the three given points. Q. E. D.

191. COROLLARY. *Two circles can intersect in only two points.*

If two circles have three points in common, can it be shown that they coincide and form one circle?

PROPOSITION XII. THEOREM

192. *The tangents to a circle drawn from an external point are equal, and make equal angles with the line joining the point to the center.*



Given PA and PB, tangents from P to the circle whose center is O, and PO the line joining P to the center O.

To prove that $PA = PB$, *and* $\angle APO = \angle OPB$.

Proof. Draw OA and OB.

PA is $\perp$ to OA, and PB is $\perp$ to OB. § 185

(*A tangent to a circle is $\perp$ to the radius drawn to the point of contact.*)

In the rt. $\triangle$ PAO and PBO,

$PO = PO$, Iden.

and $OA = OB$. § 162

$\therefore$ rt. $\triangle PAO$ is congruent to rt. $\triangle PBO$. § 89

$\therefore PA = PB$, and $\angle APO = \angle OPB$, by § 67. Q.E.D.

193. Line of Centers. The line determined by the centers of two circles is called the *line of centers.*

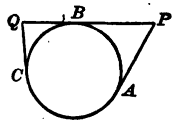
194. Tangent Circles. Two circles that are both tangent to the same line at the same point are called *tangent circles.*

Circles are said to be tangent *internally* or *externally*, according as they lie on the same side of the tangent line or on opposite sides. E.g. the two circles shown in the figure on page 110 are tangent externally.

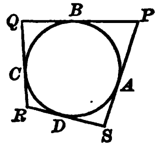
The point of contact with the line is called the *point of contact* or *point of tangency* of the circles.

EXERCISE 29

1. Show that the reasoning of § 190 will not hold for four points, and hence that a circle cannot always be drawn through four points.

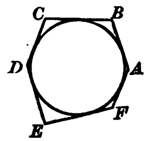


2. Tangents to a circle at A, B, C, points on the circle, meet in P and Q, as here shown. Prove that $AP + QC = PQ$.

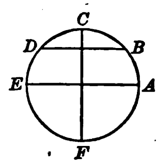


3. If a quadrilateral has each side tangent to a circle, the sum of one pair of opposite sides equals the sum of the other pair.

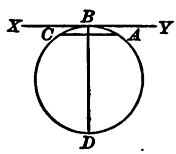
In this figure, $SP + QR = PQ + RS$.



4. The hexagon here shown has each side tangent to the circle. Prove that $AB + CD + EF = BC + DE + FA$.

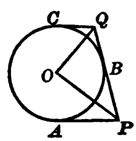


5. In this figure CF is a diameter perpendicular to the parallel chords DB and EA, and arc $AB = 40°$ and arc $BC = 50°$. How many degrees are there in arcs CD, DE, EF, and FA?



6. In this figure XY is tangent to the circle at B, the chord CA is perpendicular to the diameter BD, and the arc $CD = 150°$. How many degrees are there in arc AB?

7. If a quadrilateral has each side tangent to a circle, the sum of the angles at the center subtended by any two opposite sides is equal to a straight angle.

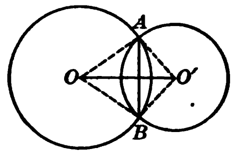


8. AP and CQ are parallel tangents meeting a third tangent QP, as shown in the figure. O being the center, prove that the angle POQ is a right angle.

Are A, O, and C in the same straight line? Draw OA and OC, and find the relations of the $\angle$s at O to those at P and Q.

PROPOSITION XIII. THEOREM

195. *If two circles intersect, the line of centers is the perpendicular bisector of their common chord.*



Given O and O', the centers of two intersecting circles, AB the common chord, and OO' the line of centers.

To prove that OO' is $\perp$ to AB at its mid-point.

Proof. Draw OA, OB, $O'A$, and $O'B$.

$OA = OB$, and $O'A = O'B$. § 162

$\therefore$ O and O' are two points, each equidistant from A and B.

$\therefore$ OO' is the perpendicular bisector of AB, by § 151. Q.E.D.

196. Common Tangents. A tangent to two circles is called a *common external tangent* if it does not cut the line-segment joining the centers, and a *common internal tangent* if it cuts it.

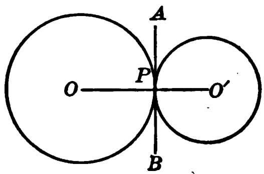
EXERCISE 30

Describe the relative position of two circles if the line-segment joining the centers is related to the radii as stated in Exs. 1–5, and illustrate each case by a figure:

1. The line-segment greater than the sum of the radii.
2. The line-segment equal to the sum of the radii.
3. The line-segment less than the sum but greater than the difference of the radii.
4. The line-segment equal to the difference of the radii.
5. The line-segment less than the difference of the radii.

PROPOSITION XIV. THEOREM

197. *If two circles are tangent to each other, the line of centers passes through the point of contact.*



Given two circles tangent at P.

To prove that P is in the line of centers.

Proof. Let AB be the common tangent at P. § 194

Then a $\perp$ to AB, drawn through the point P, passes through the centers O and O'. § 186

(*A $\perp$ to a tangent at the point of contact passes through the center of the circle.*)

Therefore the line determined by O and O', having two points in common with this $\perp$, must coincide with it. Post. 1

$\therefore P$ is in the line of centers. Q.E.D.

EXERCISE 31

Describe the relative position of two circles having tangents as stated in Exs. 1–5, and illustrate each case by a figure:

1. Two common external and two common internal tangents.

2. Two common external tangents and one common internal tangent.

3. Two common external tangents and no common internal tangent.

4. One common external and no common internal tangent.

5. No common tangent.

6. The line which passes through the mid-points of two parallel chords passes through the center of the circle.

7. If two circles are tangent externally, the tangents to them from any point of the common internal tangent are equal.

8. If two circles tangent externally are tangent to a line AB at A and B, their common internal tangent bisects AB.

9. The line drawn from the center of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord joining the points of contact.

10. The diameters of two circles are respectively 2.74 in. and 3.48 in. Find the distance between the centers of the circles if they are tangent externally. Find the distance between the centers of the circles if they are tangent internally.

11. Three circles of diameters 4.8 in., 3.6 in., and 4.2 in. are externally tangent, each to the other two. Find the perimeter of the triangle formed by joining the centers.

12. A circle of center O and radius r' rolls around a fixed circle of radius r. What is the locus of O? Prove it.

13. The line drawn from the mid-point of a chord to the mid-point of its subtended arc is perpendicular to the chord.

14. If two circles tangent externally at P are tangent to a line AB at A and B, the angle BPA is a right angle.

15. Three circles are tangent externally at the points A, B, and C, and the chords AB and AC are produced to cut the circle BC at D and E. Prove that DE is a diameter.

16. If two radii of a circle, at right angles to each other, when produced are cut by a tangent to the circle at A and B, the other tangents from A and B are parallel to each other.

17. If two common external tangents or two common internal tangents are drawn to two circles, the line-segments intercepted between the points of contact are equal.

198. Measure. The number of times a quantity of any kind contains a known unit of the same kind, expressed in terms of that known unit, is called the *measure* of the quantity.

Thus we measure the *length* of a schoolroom by finding the number of times it contains a *known unit* called the *foot.* We measure the *area* of the floor by finding the number of times it contains a *known unit* called the *square foot.* You measure your weight by finding the number of times it contains a *known unit* called the *pound.* Thus the measure of the length of a room may be 30 ft., the measure of the area of the floor may be 600 sq. ft., and so on.

The abstract number found in measuring a quantity is called its *numerical measure,* or usually simply its *measure.*

199. Ratio. The quotient of the numerical measures of two quantities, expressed in terms of a common unit, is called the *ratio* of the quantities.

Thus, if a room is 20 ft. by 35 ft., the ratio of the width to the length is 20 ft. $\div$ 35 ft., or $\frac{20}{35}$, which reduces to $\frac{4}{7}$. Here the *common unit* is 1 ft.

The ratio of a to b is written $\frac{a}{b}$, or $a : b$, as in arithmetic and algebra.

Thus the ratio of $20°$ to $30°$ is $\frac{20}{30}$, or $\frac{2}{3}$, or $2 : 3$.

200. Commensurable Magnitudes. Two quantities of the same kind that can both be expressed in integers in terms of a common unit are said to be *commensurable magnitudes.*

Thus 20 ft. and 35 ft. are expressed in integers (20 and 35) in terms of a common unit (1 ft.); similarly 2 ft. and $3\frac{1}{2}$ ft., the integers being 4 and 7, and the common unit being $\frac{1}{2}$ ft.

The common unit used in measuring two or more commensurable magnitudes is called their *common measure.* Each of the magnitudes is called a *multiple* of this common measure.

201. Incommensurable Magnitudes. Two quantities of the same kind that cannot both be expressed in integers in terms of a common unit are said to be *incommensurable magnitudes.*

Thus, if $a = \sqrt{2}$ and $b = 3$, there is no number that is contained an integral number of times in both $\sqrt{2}$ and 3. Hence a and b are, in this case, incommensurable magnitudes.

202. Incommensurable Ratio. The ratio of two incommensurable magnitudes is called an *incommensurable ratio.*

Although the exact value of such a ratio cannot be expressed by an integer, a common fraction, or a decimal fraction of a limited number of places, it may be expressed *approximately.*

Thus suppose $\frac{a}{b} = \sqrt{2}$.

Now $\sqrt{2} = 1.41421356\cdots$, which is greater than 1.414213 but less than 1.414214. Then if a *millionth part* of b is taken as the unit of measure, the value of $a : b$ lies between 1.414213 and 1.414214, and therefore differs from either by less than 0.000001.

By carrying the decimal further an approximate value may be found that will differ from the ratio by less than a *billionth,* a *trillionth,* or *any other assigned value.*

That is, *for practical purposes all ratios are commensurable.*

For example, if $\frac{a}{b} > \frac{m}{n}$ but $< \frac{m+1}{n}$, then the error in taking either of these values for $\frac{a}{b}$ is less than $\frac{1}{n}$, the difference of these ratios. But by increasing n indefinitely, $\frac{1}{n}$ can be decreased indefinitely, and a value of the ratio can be found within any required degree of accuracy.

EXERCISE 32

Find a common measure of:

1. 32 in., 24 in.
2. 48 ft., 18 ft.
3. $5\frac{1}{2}$ in., $3\frac{1}{2}$ in.
4. $2\frac{3}{4}$ lb., $1\frac{1}{2}$ lb.
5. $6\frac{1}{3}$ da., $2\frac{2}{3}$ da.
6. 14.4 in., 1.2 in.

Find the greatest common measure of:

7. 64 yd., 24 yd.
8. 51 ft., 17 ft.
9. 7.5 in., 1.25 in.
10. $3\frac{1}{3}$ in., $0.33\frac{1}{3}$ in.
11. $2\frac{3}{4}$ ft., 0.25 ft.
12. 75°, 7° 30′.

13. If $a : b = \sqrt{3}$, find an approximate value of this ratio that shall differ from the true value by less than 0.001.

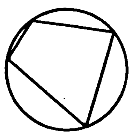
203. Constant and Variable. A quantity regarded as having a fixed value throughout a given discussion is called a *constant*, but a quantity regarded as having different successive values is called a *variable*.

204. Limit. When a variable approaches a constant in such a way that the difference between the two may become and remain less than any assigned positive quantity, however small, the constant is called the *limit* of the variable.

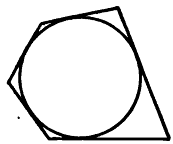
Variables can sometimes reach their limits and sometimes not. E.g. a chord may increase in length up to a certain limit, the diameter, and it can reach this limit and still be a chord; it may decrease, approaching the limit 0, but it cannot reach this limit and still be a chord.

205. Inscribed and Circumscribed Polygons. If the sides of a polygon are all chords of a circle, the polygon is said to be *inscribed* in the circle; if the sides are all tangents to a circle, the polygon is said to be *circumscribed* about the circle.

The circle is said to be *circumscribed* about the inscribed polygon, and to be *inscribed* in the circumscribed polygon.

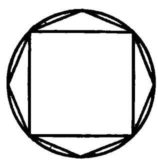


Inscribed Polygon



Circumscribed Polygon

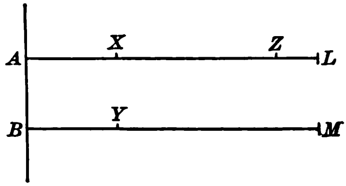
206. Circle as a Limit. If we inscribe a square in a circle, and then inscribe an octagon by taking the mid-points of the four equal arcs for the new vertices, the octagon is greater than the square but smaller than the area inclosed by the circle, and the perimeter of the octagon is greater than the perimeter of the square (§ 112).



By continually doubling the number of sides in this way it appears that the area inclosed by the circle is the limit of the area of the polygon, and the circle is the limit of its perimeter, as the number of sides is indefinitely increased.

Hence we have limiting *forms* as well as limiting *values*, the form of the circle being the limit approached by the form of the inscribed polygon.

207. Principle of Limits. *If, while approaching their respective limits, two variables are always equal, their limits are equal.*



Let AX and BY increase in length in such a way that they always remain equal, and let their respective limits be AL and BM.

To prove that $AL = BM$.

Suppose these limits are not equal, but that $AZ = BM$.

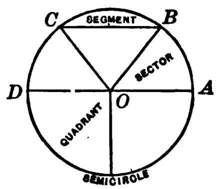
Then since X may reach a point between Z and L we may have $AX > AZ$, and therefore greater than its supposed equal, BM; but BY cannot be greater than BM. Therefore we should have $AX > BY$, which is contrary to what is given.

Hence AL cannot be greater than BM, and similarly BM cannot be greater than AL. $\therefore AL = BM$. Q.E.D.

208. Area of Circle. The area inclosed by a circle is called the *area of the circle.*

It is evident that a *diameter bisects the area of a circle.*

209. Segment. A portion of a plane bounded by an arc of a circle and its chord is called a *segment* of the circle.



If the chord is a diameter, the segment is called a *semicircle,* this word being commonly used to mean not only half of the circle but also the area inclosed by a semicircle and a diameter.

210. Sector. A portion of a plane bounded by two radii and the arc of the circle intercepted by the radii is called a *sector.*

If the arc is a quarter of the circle, the sector is called a *quadrant.*

211. Inscribed Angle. An angle whose vertex is on a circle, and whose sides are chords, is called an *inscribed angle.*

An angle is said to be *inscribed in a segment* if its vertex is on the arc of the segment and its sides pass through the ends of the arc.

Proposition XV. Theorem

212. *In the same circle or in equal circles two central angles have the same ratio as their intercepted arcs.*

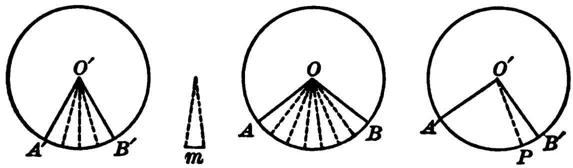


Fig. 1 Fig. 2 Fig. 3

Given two equal circles with centers O and O', AOB and $A'O'B'$ being central angles, and AB and $A'B'$ the intercepted arcs.

To prove that $$\frac{\angle A'O'B'}{\angle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}.$$

Case 1. *When the arcs are commensurable* (Figs. 1 and 2).

Proof. Let the arc m be a common measure of $A'B'$ and AB.

Apply the arc m as a measure to the arcs $A'B'$ and AB as many times as they will contain it.

Suppose m is contained a times in $A'B'$, and b times in AB.

Then $$\frac{\text{arc } A'B'}{\text{arc } AB} = \frac{a}{b}.$$

At the several points of division on AB and $A'B'$ draw radii.

These radii will divide $\angle AOB$ into b parts, and $\angle A'O'B'$ into a parts, equal each to each. § 167

$$\therefore \frac{\angle A'O'B'}{\angle AOB} = \frac{a}{b}.$$

$$\therefore \frac{\angle A'O'B'}{\angle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}, \text{ by Ax. 8.}$$ Q.E.D.

Case 2 may be omitted at the discretion of the teacher if the incommensurable cases are not to be taken in the course.

CASE 2. *When the arcs are incommensurable* (Figs. 2 and 3).

Proof. Divide AB into a number of equal parts, and apply one of these parts to $A'B'$ as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to some point, as P, leaving a remainder PB' less than one of these parts. Draw $O'P$.

By construction AB and $A'P$ are commensurable.

$$\therefore \frac{\angle A'O'P}{\angle AOB} = \frac{\text{arc } A'P}{\text{arc } AB}. \qquad \text{Case 1}$$

By increasing the *number* of equal parts into which AB is divided we can diminish the *length* of each, and therefore can make PB' less than any assigned positive value, however small.

Hence PB' approaches zero as a limit as the number of parts of AB is indefinitely increased, and at the same time the corresponding angle $PO'B'$ approaches zero as a limit. § 204

Therefore the arc $A'P$ approaches the arc $A'B'$ as a limit, and the $\angle A'O'P$ approaches the $\angle A'O'B'$ as a limit.

$\therefore$ the variable $\dfrac{\text{arc } A'P}{\text{arc } AB}$ approaches $\dfrac{\text{arc } A'B'}{\text{arc } AB}$ as a limit,

and the variable $\dfrac{\angle A'O'P}{\angle AOB}$ approaches $\dfrac{\angle A'O'B'}{\angle AOB}$ as a limit.

But $\dfrac{\angle A'O'P}{\angle AOB}$ is always equal to $\dfrac{\text{arc } A'P}{\text{arc } AB}$,

as $A'P$ varies in value and approaches $A'B'$ as a limit. Case 1

$$\therefore \frac{\angle A'O'B'}{\angle AOB} = \frac{\text{arc } A'B'}{\text{arc } AB}, \text{ by § 207.} \qquad \text{Q.E.D.}$$

213. Numerical Measure. We therefore see that the numerical measure of a central angle (in degrees, for example) equals the numerical measure of the intercepted arc. This is commonly expressed by saying that *a central angle is measured by the intercepted arc.*

PROPOSITION XVI. THEOREM

214. *An inscribed angle is measured by half the intercepted arc.*

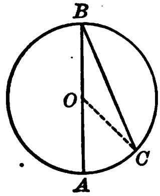


FIG. 1

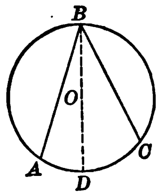


FIG. 2

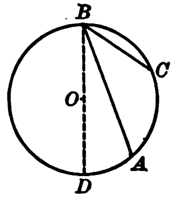


FIG. 3

Given a circle with center O and the inscribed angle B, intercepting the arc AC.

To prove that $\angle B$ is measured by half the arc AC.

CASE 1. *When O is on one side, as AB* (Fig. 1).

Proof. Draw OC.

Then	$\because OC = OB,$	§ 162
	$\therefore \angle B = \angle C.$	§ 74
But	$\angle B + \angle C = \angle AOC.$	§ 111
	$\therefore 2\angle B = \angle AOC.$	Ax. 9
	$\therefore \angle B = \frac{1}{2} \angle AOC.$	Ax. 4
But	$\angle AOC$ is measured by arc AC.	§ 213
	$\therefore \frac{1}{2} \angle AOC$ is measured by $\frac{1}{2}$ arc AC.	Ax. 4
	$\therefore \angle B$ is measured by $\frac{1}{2}$ arc AC.	Ax. 9

CASE 2. *When O lies within the angle B* (Fig. 2).

Proof. Draw the diameter BD.

Then $\angle ABD$ is measured by $\frac{1}{2}$ arc AD,

and $\angle DBC$ is measured by $\frac{1}{2}$ arc DC. Case 1

$\therefore \angle ABD + \angle DBC$ is measured by $\frac{1}{2}$ (arc AD + arc DC),

or $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

CASE 3. *When O lies outside the angle B* (Fig. 3).

Proof. Draw the diameter BD.

Then $\angle DBC$ is measured by $\frac{1}{2}$ arc DC,

and $\angle DBA$ is measured by $\frac{1}{2}$ arc DA. Case 1

$\therefore \angle DBC - \angle DBA$ is measured by $\frac{1}{2}$ (arc DC − arc DA),

or $\angle ABC$ is measured by $\frac{1}{2}$ arc AC. Q.E.D.

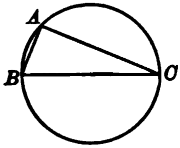


FIG. 4

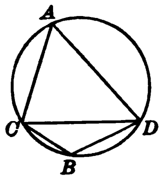


FIG. 5

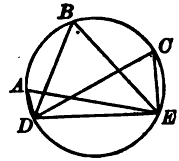


FIG. 6

215. COROLLARY 1. *An angle inscribed in a semicircle is a right angle.*

For it is half of a central straight angle, as in Fig. 4.

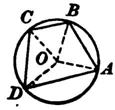
216. COROLLARY 2. *An angle inscribed in a segment greater than a semicircle is an acute angle, and an angle inscribed in a segment less than a semicircle is an obtuse angle.*

See $\angle s$ A and B in Fig. 5.

217. COROLLARY 3. *Angles inscribed in the same segment or in equal segments are equal.*

Why is this? (Fig. 6.)

218. COROLLARY 4. *If a quadrilateral is inscribed in a circle, the opposite angles are supplementary; and, conversely, if two opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*



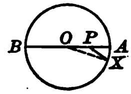
For the second part, can a circle be passed through A, B, C (§ 190)? If it does not pass through D also, can you show that $\angle D$ would be greater than or less than some other angle (§ 111) that is supplementary to $\angle B$?

EXERCISE 33

1. A parallelogram inscribed in a circle is a rectangle.

2. A trapezoid inscribed in a circle is isosceles.

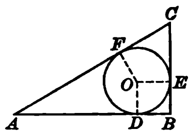
3. The shorter segment of the diameter through a given point within a circle is the shortest line that can be drawn from that point to the circle.



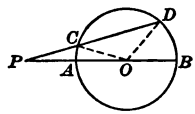
Let P be the given point. Prove PA shorter than any other line PX from P to the circle.

4. The longer segment of the diameter through a given point within a circle is the longest line that can be drawn from that point to the circle.

5. The diameter of the circle inscribed in a right triangle is equal to the difference between the hypotenuse and the sum of the other two sides.



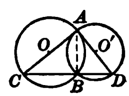
6. A line from a given point outside a circle passing through the center contains the shortest line-segment that can be drawn from that point to the circle.



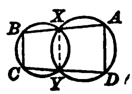
Let P be the point, O the center, A the point where PO cuts the circle, and C any other point on the circle. How does $PC+CO$ compare with PO?

7. A line from a given point outside a circle passing through the center contains the longest line-segment (to the concave arc) that can be drawn from that point to the circle.

8. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.

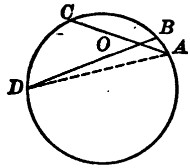


9. If two circles intersect and a line is drawn through each point of intersection terminated by the circles, the chords joining the corresponding ends of these lines are parallel.



PROPOSITION XVII. THEOREM

219. *An angle formed by two chords intersecting within the circle is measured by half the sum of the intercepted arcs.*



Given the angle AOB formed by the chords AC and BD.

To prove that $\angle AOB$ is measured by $\frac{1}{2}$ (arc AB + arc CD).

Proof. Draw AD.

Then $\angle AOB = \angle A + \angle D$. § 111

(*An exterior $\angle$ of a $\triangle$ is equal to the sum of the two opposite interior* $\angle$s.)

But $\angle A$ is measured by $\frac{1}{2}$ arc CD, § 214

(*An inscribed $\angle$ is measured by half the intercepted arc.*)

and $\angle D$ is measured by $\frac{1}{2}$ arc AB. § 214

$\therefore \angle AOB$ is measured by $\frac{1}{2}$(arc AB + arc CD), by Ax. 1. Q.E.D.

Discussion. If O is at the center of the circle, to what previous proposition does this proposition reduce?

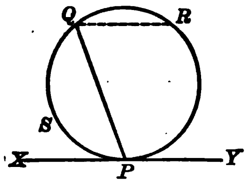
If O is on the circle, as at B, to what previous proposition does this proposition reduce?

Suppose the point O remains as in the figure, and the chord AC swings about O as a pivot until it coincides with the chord BD. What can then be said of the measure of $\angle$s AOB and COD? What can be said as to the measure of $\angle$s BOC and DOA?

It is also possible to prove the proposition by drawing a chord AE parallel to BD, and showing that $\angle AOB = \angle A$, since they are alternate-interior angles formed by a transversal cutting two parallels. Now $\angle A$ is measured by $\frac{1}{2}$ arc CE. But arc CE = arc CD + arc DE, or arc CD + arc AB, since arc AB = arc DE (§ 189). Therefore $\angle AOB$, which equals $\angle A$, is measured by $\frac{1}{2}$ (arc AB + arc CD).

Proposition XVIII. Theorem

220. *An angle formed by a tangent and a chord drawn from the point of contact is measured by half the intercepted arc.*



Given the chord PQ and the tangent XY through P.

To prove that $\angle QPX$ *is measured by half the arc QSP.*

Proof. Suppose the chord QR drawn from the point Q parallel to the tangent XY.

Then arc $PR =$ arc QSP. § 189

(*Two parallel lines intercept equal arcs on a circle.*)

Also $\angle QPX = \angle PQR$. § 100

(*If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*)

But $\angle PQR$ is measured by $\frac{1}{2}$ arc PR. § 214

(*An inscribed ∠ is measured by half the intercepted arc.*)

Substitute $\angle QPX$ for its equal, the $\angle PQR$,

and substitute arc QSP for its equal, the arc PR.

Then $\angle QPX$ is measured by $\frac{1}{2}$ arc QSP, by Ax. 9. Q.E.D.

Discussion. By half of what arc is $\angle YPQ$, the supplement of $\angle QPX$, measured?

If PQ should be drawn so as to be perpendicular to XY, by what would ∠s YPQ and QPX be measured?

Suppose PQ swings about the point P as a pivot until it coincides with XY, by what will $\angle YPQ$ be measured? By what will $\angle QPX$ be measured, and what will it equal?

PROPOSITION XIX. THEOREM

221. *An angle formed by two secants, a secant and a tangent, or two tangents, drawn to a circle from an external point, is measured by half the difference of the intercepted arcs.*

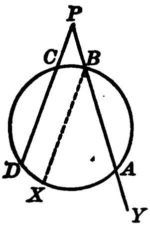


FIG. 1

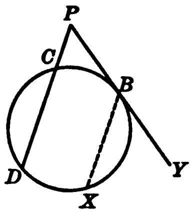


FIG. 2

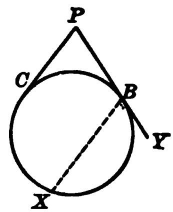


FIG. 3

Given two secants *PBA* and *PCD*, from the external point *P*.

To prove that $\angle P$ *is measured by* $\frac{1}{2}$ *(arc DA – arc BC).*

Proof. Suppose the chord BX drawn ∥ to PCD (Fig. 1).

Then $\qquad$ arc BC = arc DX. $\qquad$ § 189

Furthermore $\qquad$ arc XA = arc DA – arc DX.

$\therefore$ arc XA = arc DA – arc BC. $\qquad$ Ax. 9

Also $\qquad$ $\angle P = \angle XBA$. $\qquad$ § 102

But $\angle XBA$ is measured by $\frac{1}{2}$ arc XA. $\qquad$ § 214

Substitute $\angle P$ for its equal, the $\angle XBA$,

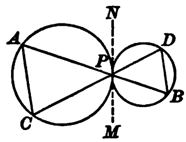
and substitute arc DA – arc BC for its equal, the arc XA.

Then $\angle P$ is measured by $\frac{1}{2}$ (arc DA – arc BC), by Ax. 9. Q.E.D.

If the secant $PBAY$ swings around to tangency, it becomes the tangent PB and Fig. 1 becomes Fig. 2. If the secant PCD also swings around to tangency, it becomes the tangent PC and Fig. 2 becomes Fig. 3. The proof of the theorem for each of these cases is left for the student.

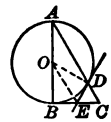
EXERCISE 34

1. If two circles touch each other and two lines are drawn through the point of contact terminated by the circles, the chords joining the ends of these lines are parallel.



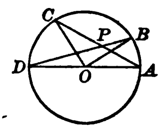
This could be proved if it could be shown that $\angle A$ equals what angle? To what two angles can these angles be proved equal by § 220? Are those angles equal?

2. If one side of a right triangle is the diameter of a circle, the tangent at the point where the circle cuts the hypotenuse bisects the other side.

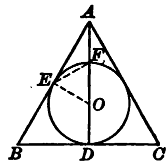


If OE is ∥ to AC, then because $BO = OA$, what is the relation of BE to EC? The proposition therefore reduces to proving that OE is parallel to what line? This can be proved if $\angle BOE$ can be shown equal to what angle?

3. If from the extremities of a diameter AD two chords, AC and DB, are drawn intersecting at P so as to make $\angle APB = 45°$, then $\angle BOC$ is a right angle.

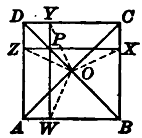


4. The radius of the circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

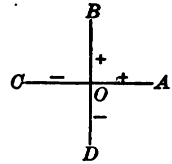


To prove this we must show that AF equals what line? It looks as if AF might equal EF, and EF equal OF. Is there any way of proving $\triangle OFE$ equilateral? of proving $\triangle AEF$ isosceles?

5. If two lines are drawn through any point in a diagonal of a square parallel to the sides, the points where these lines meet the sides lie on the circle whose center is the point of intersection of the diagonals.

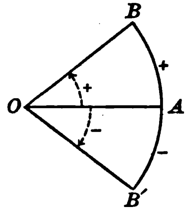


$OY = OZ$ if what two ▲ are congruent? Why are these ▲ congruent? $OY = OX$ if what two ▲ are congruent? $OX = OW$ if what two ▲ are congruent?



222. Positive and Negative Quantities. In geometry, as in algebra, quantities may be distinguished as *positive* and as *negative*.

Thus as we consider temperature above zero positive and temperature below zero negative, so in this figure, if OB is considered *positive*, then OD may be considered *negative*. Similarly, if OA is considered positive, then OC may be considered negative.



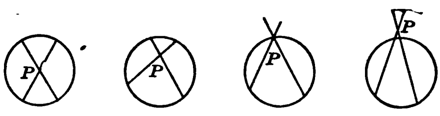
Likewise with respect to angles and arcs, if the rotating line OA moves in the direction of AB, counterclockwise, the angle and arc generated are considered *positive*. If it rotates in the direction AB', like the hands of a clock, the angle and arc generated are considered *negative*.

223. Principle of Continuity. By considering the distinction between positive and negative magnitudes, a theorem may often be so stated as to include several particular theorems. For example, *The angle included between two lines that cut or are tangent to a circle is measured by half the sum of the intercepted arcs.*

In particular: 1. If the lines intersect at the center, half the sum of the arcs will then become simply one of the arcs, and the proposition reduces to that of § 213.

2. If the lines are two general chords, we have the case of § 219.

3. If the point of intersection P moves to the circle, we have the case of § 214, one arc becoming zero.



4. If P moves outside the circle, then the smaller arc passes through zero and becomes negative, so that the sum of the arcs becomes their arithmetical difference (§ 221).

We may continue the discussion so as to include all the cases of the propositions proved from § 213 to § 221.

When the reasoning employed to prove a theorem is continued as just illustrated, so as to include several theorems, we are said to reason by the *Principle of Continuity*.

224. Problems of Construction. At the beginning of the study of geometry some directions were given for simple constructions, so that figures might be drawn with accuracy. It was not proved at that time that these constructions were correct, because no theorems had been studied on which proofs could be based. It is now purposed to review these constructions, to prove that they are correct, and to apply the methods employed to the solution of more difficult problems.

225. Nature of a Solution. A solution of a problem has one requirement that a proof of a theorem does not have.

In a theorem we have three general steps: (1) *Given*, (2) *To prove*, (3) *Proof*. In a problem we have four steps: (1) *Given*, (2) *Required* (to do some definite thing), (3) *Construction* (showing how to do it), (4) *Proof* (that the construction is correct).

We *prove* a theorem, but we *solve* a problem, and then prove that our solution is a correct one.

In the figures of this text given lines are shown as full, black lines; construction lines and lines required are shown as dotted lines.

226. Discussion of a Problem. Besides the four necessary general steps in treating a problem, there is a desirable step to be taken in many cases. This is the discussion of the problem, in which is considered whether there is more than one solution, and other similar questions.

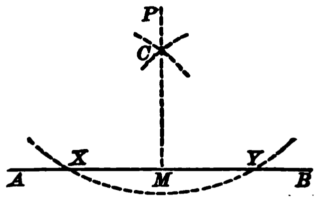
For example, suppose the problem is this: *Required from a given point to draw a tangent to a circle.*

After the problem has been solved we may discuss it thus: In general, if the given point is outside the circle, two tangents may be drawn, and these tangents are equal (§ 192); if the given point is on the circle only one tangent can be drawn, since only one perpendicular can be drawn to a radius at its extremity (§ 184); if the given point is within the circle, evidently no tangent can be drawn.

In the discussion the Principle of Continuity often enters, the figure being studied for various positions of some given point or line, as was done in the discussions on pages 121 and 122.

PROPOSITION XX. PROBLEM

227. *To let fall a perpendicular upon a given line from a given external point.*



Given the line *AB* and the external point *P*.

Required from P to let fall a ⊥ upon AB.

Construction. With P as a center, and a radius sufficiently great, describe an arc cutting AB at X and Y. Post. 4

With X and Y as centers, and a convenient radius, describe two arcs intersecting at C. Post. 4

Draw PC. Post. 1

Produce PC to intersect AB at M. Post. 2

Then PM is the line required. Q.E.F.

Proof. Since P and C are by construction two points each equidistant from X and Y, they determine the perpendicular to XY at its mid-point. § 151

(*Two points each equidistant from the extremities of a line determine the ⊥ bisector of the line.*) Q.E.D.

Discussion. The following are interesting considerations:

That PC produced will really intersect AB, as stated in the construction, is shown in the proof.

A convenient radius to take for the two intersecting arcs is XY.

If C falls on P, take C at the other intersection of the arcs below AB, as is seen in the figure of Ex. 2, p. 9.

To obtain a radius for the first circle, draw any line from P that will cut AB, and use that.

Proposition XXI. Problem

228. *At a given point in a given line, to erect a perpendicular to the line.*

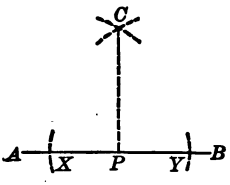


Fig. 1

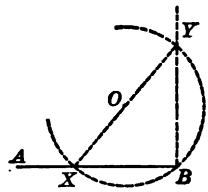


Fig. 2

Given the point *P* in the line *AB*.

Required to erect a $\perp$ *to AB at P.*

Case 1. *When the point P is not at the end of AB* (Fig. 1).

Construction. Take $PX = PY$. Post. 4

With X and Y as centers, and a convenient radius, describe arcs intersecting at C. Post. 4

Draw CP. Post. 1

Then CP is $\perp$ to AB. Q.E.F.

Proof. P and C, two points each equidistant from X and Y, determine the $\perp$ bisector of XY, by § 151. Q.E.D.

Case 2. *When the point P is at the end of AB* (Fig. 2).

Construction. Suppose P to coincide with B.

Take any point O outside of AB, and with O as a center and OB as a radius describe a circle intersecting AB at X.

From X draw the diameter XY, and draw BY. Post. 1

Then BY is $\perp$ to AB. Q.E.F.

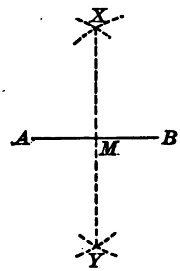
Proof. $\angle B$ is a right angle. § 215

$\therefore BY$ is $\perp$ to AB, by § 27. Q.E.D.

Discussion. If the circle described with O as a center is tangent to AB at B, then OB is the required perpendicular (§ 185).

Proposition XXII. Problem

229. *To bisect a given line.*



Given the line AB.

Required to bisect AB.

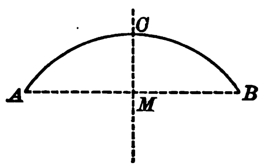
Construction. With A and B as centers and AB as a radius describe arcs intersecting at X and Y, and draw XY. Post. 4

Then XY bisects AB. Q. E. F.

Proof. XY bisects AB, by § 151. Q. E. D.

Proposition XXIII. Problem

230. *To bisect a given arc.*



Given the arc AB.

Required to bisect AB.

Construction. Draw the chord AB. Post. 1

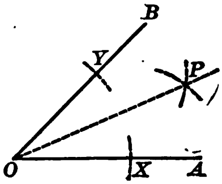
Draw CM, the perpendicular bisector of the chord AB. § 229

Then CM bisects the arc AB. Q. E. F.

Proof. CM bisects the arc AB, by § 177. Q. E. D.

Proposition XXIV. Problem

231. *To bisect a given angle.*



Given the angle AOB.

Required to bisect $\angle AOB$.

Construction. With O as a center and any radius describe an arc cutting OA at X and OB at Y. Post. 4

With X and Y as centers and XY as a radius describe arcs intersecting at P. Post. 4

Draw OP. Post. 1

Then OP bisects $\angle AOB$. Q.E.F.

Proof. Draw PX and PY.

Then prove that the $\triangle$ OXP and OYP are congruent. § 80

Then $\angle AOP = \angle POB$, by § 67. Q.E.D.

EXERCISE 35

1. To construct an angle of 45°; of 135°.

2. To construct an angle of 22° 30′; of 157° 30′.

3. To construct an equilateral triangle, having given one side, and thus to construct an angle of 60°.

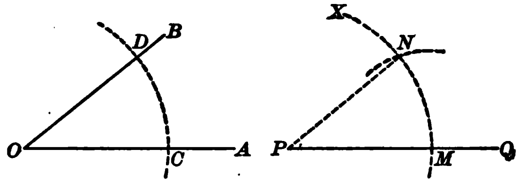
4. To construct an angle of 30°; and thus to trisect a right angle.

5. To construct an angle of 15°; of 7° 30′; of 195°; of 345°.

6. To construct a triangle having two of its angles equal to 75°. Is the triangle definitely determined?

PROPOSITION XXV. PROBLEM

232. *From a given point in a given line, to draw a line making an angle equal to a given angle.*



Given the angle AOB and the point P in the line PQ.

Required from P to draw a line making with the line PQ an angle equal to $\angle AOB$.

Construction. With O as a center and any radius describe an arc cutting OA at C and OB at D. Post. 4

With P as a center and the same radius describe an arc MX, cutting PQ at M. Post. 4

With M as a center and a line joining C and D as a radius describe an arc cutting the arc MX at N. Post. 4

Draw PN. Post. 1

Then $\angle QPN = \angle AOB$. Q.E.F.

Proof. Draw CD and MN.

Then prove that the $\triangle$ PMN and OCD are congruent. § 80

Then $\angle QPN = \angle AOB$, by § 67. Q.E.D.

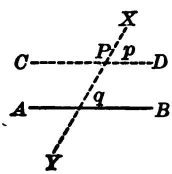
233. COROLLARY. *Through a given external point, to draw a line parallel to a given line.*

Let AB be the given line and P the given external point.

Draw any line XPY through P, cutting AB as in the figure.

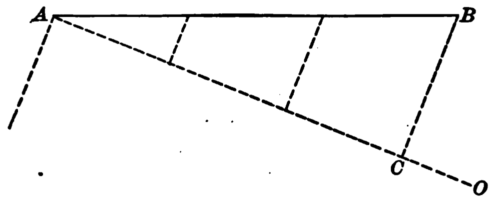
Draw CD through P, making $\angle p = \angle q$.

The line CD will be the line required.



Proposition XXVI. Problem

234. *To divide a given line into a given number of equal parts.*



Given the line AB.

Required to divide AB into a given number of equal parts.

Construction. From A draw the line AO, making any convenient angle with AB. Post. 1

Take any convenient length, and by describing arcs apply it to AO as many times as is indicated by the number of parts into which AB is to be divided. Post. 4

From C, the last point thus found on AO, draw CB. Post. 1

From the division points on AO draw parallels to CB. § 233

These lines divide AB into equal parts. Q.E.F.

Proof. These lines divide AB into equal parts, by § 134. Q.E.D.

EXERCISE 36

1. To divide a given line into four equal parts.

2. To construct an equilateral triangle, given the perimeter.

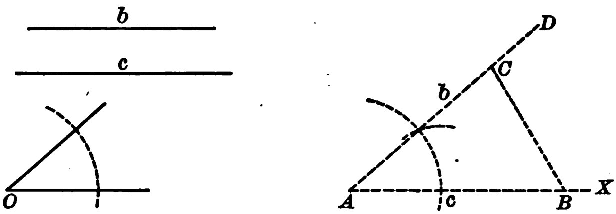
3. Through a given point, to draw a line which shall make equal angles with the two sides of a given angle.

4. Through a given point, to draw two lines so that they shall form with two intersecting lines two isosceles triangles.

5. To construct a triangle having its three angles respectively equal to the three angles of a given triangle.

PROPOSITION XXVII. PROBLEM

235. *To construct a triangle when two sides and the included angle are given.*



Given b and c two sides of a triangle, and O the included angle,

Required to construct the triangle.

Construction. On any line as AX, by describing an arc, mark off AB equal to c. Post. 4

At A construct $\angle BAD$ equal to $\angle O$. § 232

On AD, by describing an arc, mark off AC equal to b. Post. 4

Draw BC. Post. 1

Then $\triangle ABC$ is the $\triangle$ required. Q.E.F.

Proof. (Left for the student.)

236. COROLLARY 1. *To construct a triangle when a side and two angles are given.*

There are two cases to be considered: (1) when the given side is included between the given angles; and (2) when it is not (in which case find the other angle by § 107).

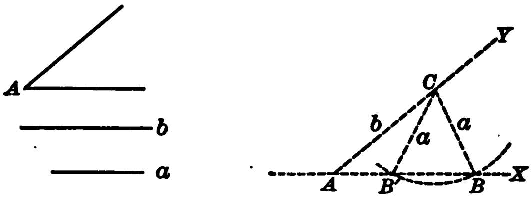
237. COROLLARY 2. *To construct a triangle when the three sides are given.*

238. COROLLARY 3. *To construct a parallelogram when two sides and the included angle are given.*

Combine § 235 and § 233.

Proposition XXVIII. Problem

239. *To construct a triangle when two sides and the angle opposite one of them are given.*



Given a and b two sides of a triangle, and A the angle opposite a.

Required to construct the triangle.

Construction. Case 1. *If a is less than b.*

Construct $\angle XAY$ equal to the given $\angle A$. § 232

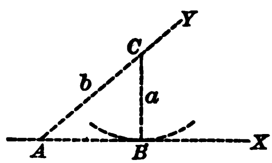
On AY take AC equal to b.

From C as a center, with a radius equal to a, describe an arc intersecting the line AX at B and B'.

Draw BC and $B'C$, thus completing the triangle.

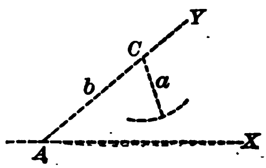
Then both the $\triangle ABC$ and $AB'C$ satisfy the conditions, and hence we have two constructions. Q.E.F.

This is called the *ambiguous* case.



Discussion. If the given side a is equal to the $\perp$ CB, the arc described from C will touch AX, and there will be but one construction, the rt. $\triangle ABC$.

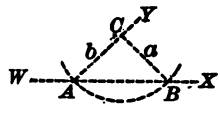
If the given side a is less than the perpendicular from C, the arc described from C will not intersect or touch AX, and hence a construction is impossible.



If $\angle A$ is right or obtuse, a construction is impossible, since $a < b$; for the side of a triangle opposite a right or obtuse angle is the longest side (§ 114).

CASE 2. *If a is equal to b.*

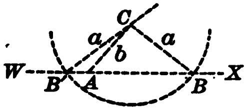
If the given $\angle A$ is acute, and $a = b$, the arc described from C as a center, and with a radius equal to a, will cut the line WX at the points A and B. There is therefore but one triangle that satisfies the conditions, namely the isosceles $\triangle ABC$.



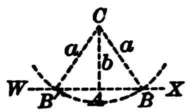
Discussion. If the $\angle A$ is right or obtuse, a construction is impossible when $a = b$; for equal sides of a triangle have equal angles opposite them, and a triangle cannot have two right angles or two obtuse angles (§ 109).

CASE 3. *If a is greater than b.*

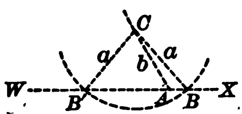
If the given $\angle A$ is acute, the arc described from C will cut the line WX on opposite sides of A, at B and B'. The $\triangle ABC$ satisfies the conditions, but the $\triangle AB'C$ does not, for it does not contain the acute $\angle A$. There is then only one triangle that satisfies the conditions, namely the $\triangle ABC$.



If the given $\angle A$ is right, the arc described from C cuts the line WX on opposite sides of A at the points B and B', and we have the two *congruent* right triangles ABC and $AB'C$ that satisfy the conditions.



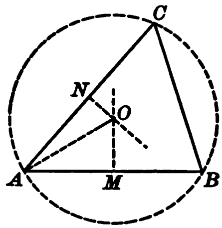
If the given $\angle A$ is obtuse, the arc described from C cuts the line WX on opposite sides of A, at the points B and B'. The $\triangle ABC$ satisfies the conditions, but the $\triangle AB'C$ does not, for it does not contain the obtuse $\angle A$. There is then only one triangle that satisfies the conditions, namely the $\triangle ABC$.



Discussion. We therefore see that when $a > b$, we have only one triangle that satisfies the conditions, for the two congruent right triangles give us only one distinct triangle.

Proposition XXIX. Problem

240. *To circumscribe a circle about a given triangle.*



Given the triangle ***ABC.***

Required to circumscribe a ⊙ about △ ABC.

Construction. Draw the perpendicular bisectors of the sides AB and AC. § 229

Since AB is not the prolongation of CA, these ⊥s will intersect at some point O. Otherwise they would be ∥, and one of them would have to be ⊥ to two intersecting lines. § 82

With O as a center, and a radius OA, describe a circle. Post. 4

The ⊙ ABC is the ⊙ required. Q.E.F.

Proof. The point O is equidistant from A and B, and also is equidistant from A and C. § 150

∴ the point O is equidistant from A, B, and C.

∴ a ⊙ described with O as a center, with a radius equal to OA, will pass through the vertices A, B, and C, by § 160. Q.E.D.

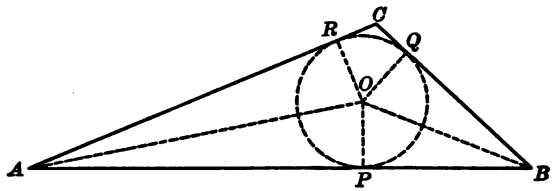
241. Corollary 1. *To describe a circle through three points not in the same straight line.*

242. Corollary 2. *To find the center of a given circle or of the circle of which an arc is given.*

243. Circumcenter. The center of the circle circumscribed about a polygon is called the *circumcenter* of the polygon.

PROPOSITION XXX. PROBLEM

244. *To inscribe a circle in a given triangle.*



Given the triangle ***ABC.***

Required to inscribe a ⊙ *in* △ *ABC.*

Construction. Bisect the ∠s *A* and *B*. § 231

From *O*, the intersection of the bisectors, draw *OP* ⊥ to the side *AB*. § 227

With *O* as a center and a radius *OP*, describe the ⊙ *PQR*.

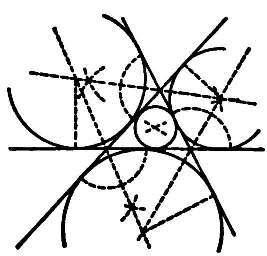
The ⊙ *PQR* is the ⊙ required. Q.E.F.

Proof. Since *O* is in the bisector of the ∠ *A*, it is equidistant from the sides *AB* and *AC*; and since *O* is in the bisector of the ∠ *B*, it is equidistant from the sides *AB* and *BC*. § 152

∴ a circle described with *O* as a center, and a radius *OP*, will touch the sides of the triangle, by § 184. Q.E.D.

245. Incenters and Excenters. The center of a circle inscribed in a polygon is called the *incenter* of the polygon.

The intersections of the bisectors of the exterior angles of a triangle are the centers of three circles, each tangent to one side of the triangle and the two other sides produced. These three circles are called *escribed* circles; and their centers are called the *excenters* of the triangle.



Proposition XXXI. Problem

246. *Through a given point, to draw a tangent to a given circle.*

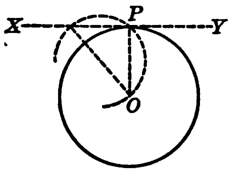


Fig. 1

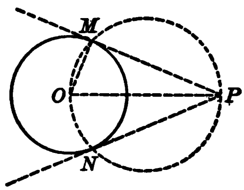


Fig. 2

Given the point *P* and the circle with center *O*.

Required through P to draw a tangent to the circle.

Case 1. *When the given point is on the circle* (Fig. 1).

Construction. From the center O draw the radius OP. Post. 1

Through P draw $XY \perp$ to OP. § 228

Then XY is the tangent required. Q.E.F.

Proof. Since XY is $\perp$ to the radius OP, Const.

$\therefore XY$ is tangent to the $\odot$ at P, by § 184. Q.E.D.

Case 2. *When the given point is outside the circle* (Fig. 2).

Construction. Draw OP. Post. 1

Bisect OP. § 229

With the mid-point of OP as a center and a radius equal to $\frac{1}{2}$ OP, describe a circle intersecting the given circle at the points M and N, and draw PM.

Then PM is the tangent required. Q.E.F.

Proof. Draw OM.

$\angle OMP$ is a right angle. § 215

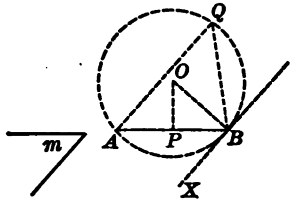
$\therefore PM$ is $\perp$ to OM. § 27

$\therefore PM$ is tangent to the circle at M, by § 184. Q.E.D.

Discussion. In like manner, we may prove PN tangent to the given $\odot$.

PROPOSITION XXXII. PROBLEM

247. *Upon a given line as a chord, to describe a segment of a circle in which a given angle may be inscribed.*



Given the line AB and the angle m.

Required on AB as a chord, to describe a segment of a circle in which $\angle m$ may be inscribed.

Construction. Construct the $\angle ABX$ equal to the $\angle m$. § 232

Bisect the line AB by the $\perp$ PO. § 229

From the point B draw BO $\perp$ to XB. § 228

With O, the point of intersection of PO and BO, as a center, and a radius equal to OB, describe a circle.

The segment BQA is the segment required. Q.E.F.

Proof. The point O is equidistant from A and B. § 150

$\therefore$ the circle will pass through A and B. § 160

But BX is $\perp$ to OB. Const.

$\therefore$ BX is tangent to the $\odot$. § 184

$\therefore$ $\angle ABX$ is measured by $\frac{1}{2}$ arc AB. § 220

But any angle, as the $\angle Q$, inscribed in the segment ABQ is measured by $\frac{1}{2}$ arc AB. § 214

$\therefore \angle Q = \angle ABX$. Ax. 8

But $\angle ABX = \angle m$. Const.

$\therefore$ $\angle m$ may be inscribed in the segment BQA, by § 217. Q.E.D.

248. How to attack a Problem. There are three common methods by which to attack a new problem:

(1) By synthesis;
(2) By analysis;
(3) By the intersection of loci.

249. Synthetic Method. If a problem is so simple that the solution is obvious from a known proposition, we have only to make the construction according to the proposition, and then to give the synthetic proof, if a proof is necessary, that the construction is correct.

It is rarely the case, however, that a problem is so simple as to allow this method to be used. We therefore commonly resort at once to the second method.

250. Analytic Method. This is the usual method of attack, and is as follows:

(1) Suppose the problem solved and see what results follow.

(2) Then see if it is possible to attain these results and thus effect the required construction; in other words, try to work backwards.

The third method, by the intersection of loci, is considered on page 143.

251. Determinate, Indeterminate, and Impossible Cases. A problem that has a definite number of solutions is said to be *determinate.* A problem that has an indefinite number of solutions is said to be *indeterminate.* A problem that has no solution is said to be *impossible.*

For example, to construct a triangle, having given its sides, is determinate; to construct a quadrilateral, having given its sides, is indeterminate; to construct a triangle with sides 2 in., 3 in., and 6 in. is impossible.

252. Discussion. The examination of a problem with reference to all possible conditions, particularly with respect to the number of solutions, is called the *discussion* of the problem.

Discussions have been given in several of the preceding problems.

253. Applications of the Analytic Method. The following are examples of the use of analysis in the solution of problems.

EXERCISE 37

1. In a triangle ABC, to draw PQ parallel to the base AB, cutting the sides in P and Q, so that PQ shall equal $AP + BQ$.

Analysis. Assume the problem solved.

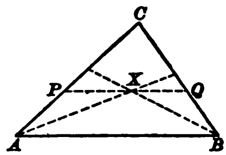
Then AP must equal some part of PQ, as PX, and BQ must equal QX.

But if $AP = PX$, what must $\angle PXA$ equal?

$\because PQ$ is $\parallel$ to AB, what does $\angle PXA$ equal?

Then why must $\angle BAX = \angle XAP$?

Similarly, what about $\angle QBX$ and $\angle XBA$?



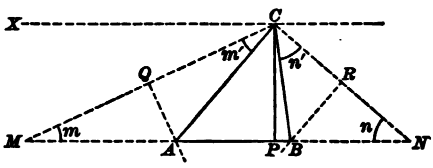
Construction. Now reverse the process. What should we do to $\angle A$ and B in order to fix X? Then how shall PQ be drawn? Now give the proof.

2. To construct a triangle, having given the perimeter, one angle, and the altitude from the vertex of the given angle.

Analysis. Let ABC be the triangle, $\angle C$ the given angle, and CP the given altitude, and assume that the problem is solved.

Since the perimeter is given as a definite line, we now try producing AB and BA, making $BN = BC$, and $AM = AC$.

Then $\angle m =$ what angle, and $\angle n =$ what angle?



Then $\quad \angle m + \angle n + \angle MCN = 180°.$

But $\quad \angle MCN = \angle m' + \angle ACB + \angle n'.$

$\therefore 2\angle m + 2\angle n + \angle ACB = 180°.$ (Why?)

$\therefore \angle m + \angle n + \frac{1}{2}\angle ACB = 90°,$

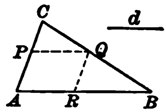
or $\quad \angle m + \angle n = 90° - \frac{1}{2}\angle ACB.$

$\therefore \angle MCN = 90° + \frac{1}{2}\angle ACB.$ (Why?)

$\therefore \angle MCN$ is known.

Construction. Now reverse the process. Draw MN equal to the perimeter. Then on MN construct a segment in which $\angle MCN$ may be inscribed (§ 247). Draw $XC \parallel$ to MN at the distance CP from MN, cutting the arc at C. Then A and B are on the $\perp$ bisectors of CM and CN. Why?

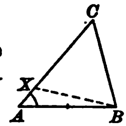
3. To draw through two sides of a triangle a line parallel to the third side, so that the part intercepted between the sides shall have a given length.



If $PQ = d$, what does AR equal? How will you reverse the reasoning?

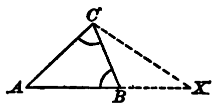
4. To draw a tangent to a given circle so that it shall be parallel to a given line.

5. To construct a triangle, having given a side, an adjacent angle, and the difference of the other sides.



If AB, $\angle A$, and $AC - BC$ are known, what points are determined? Then can XB be drawn? What kind of a triangle is $\triangle XBC$? How can C be located?

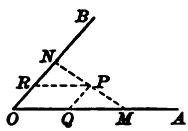
6. To construct a triangle, having given two angles and the sum of two sides.



Can the third $\angle$ be found? Assume the problem solved. If $AX = AB + BC$, what kind of a triangle is $\triangle BXC$? What does $\angle CBA$ equal? Is $\angle X$ known? How can C be fixed?

7. To construct a square, having given the diagonal.

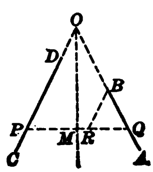
8. To draw through a given point P between the sides of an angle AOB a line terminated by the sides of the angle and bisected at P.



If $PM = PN$, and PR is $\parallel$ to AO, what can you say as to OR and RN? Can you now reverse this? Similarly, if PQ is $\parallel$ to BO, is $OQ =$ to QM?

9. To draw a line that would bisect the angle formed by two lines if those lines were produced to meet.

If AB and CD are the given lines, consider what would be the conditions if they could be produced to meet at O. Then the bisector of $\angle O$ would be the $\perp$ bisector of PQ, a line drawn so as to make equal angles with the two given lines.

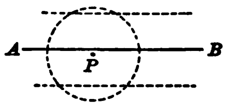


Now reverse this. How can we draw PQ so as to make $\angle P = \angle Q$? Draw $BR \parallel$ to DC, and lay off $BR = BQ$. Then draw QRP and prove that this is such a line. Then draw its $\perp$ bisector.

254. Intersection of Loci. The third general method of attack mentioned in § 248 is by intersection of loci. This is very convenient when we wish to find a point satisfying two conditions, each of which involves some locus.

EXERCISE 38

1. To find a point that is $\frac{1}{4}$ in. from a given point and $\frac{3}{16}$ in. from a given line.



If P is the given point, what is the locus of a point $\frac{1}{4}$ in. from P? If AB is the given line, what is the locus of a point $\frac{3}{16}$ in. from AB? These two loci intersect in how many points at most? Discuss the solution.

2. To find a point that is $\frac{1}{2}$ in. from one given point and $\frac{3}{4}$ in. from another given point.

Discuss the number of possible points answering the conditions.

3. To find a point that is $\frac{1}{2}$ in. from the vertex of an angle and equidistant from the sides of the angle.

4. To find a point that is equidistant from two intersecting lines and $\frac{1}{4}$ in. from their point of intersection.

How many such points can always be found?

5. To find a point that is $\frac{1}{2}$ in. from a given point and equidistant from two intersecting lines.

Discuss the problem for various positions of the given point.

6. To find a point that is $\frac{1}{2}$ in. from a given point and equidistant from two parallel lines.

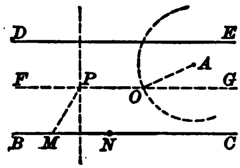
Discuss the problem for various positions of the given point.

7. Find the locus of the mid-point of a chord of a given length that can be drawn in a given circle.

8. Find the locus of the mid-point of a chord drawn through a given point within a given circle.

9. To describe a circle that shall pass through a given point and cut equal chords of a given length from two parallels.

Analysis. Let A be the given point, BC and DE the given parallels, MN the given length, and O the center of the required circle.

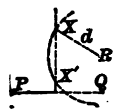


Since the circle cuts equal chords from two parallels, what must be the relative distance of its center from each? Therefore what line must be one locus for O?

Draw the $\perp$ bisector of MN, cutting FG at P. How, then, does PM compare with the radius of the circle required? How shall we then find a point O on FG that is at a distance PM from A? Do we then know that O is the center of the required circle?

10. To describe a circle that shall be tangent to each of two given intersecting lines.

11. To find in a given line a point that is equidistant from two given points.



12. To find a point that is equidistant from two given points and at a given distance from a third given point.

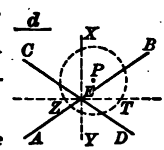
13. To describe a circle that has a given radius and passes through two given points.

14. To find a point at given distances from two given points.

15. To describe a circle that has its center in a given line and passes through two given points.

16. To find a point that is equidistant from two given points and also equidistant from two given intersecting lines.

17. To find a point that is equidistant from two given points and also equidistant from two given parallel lines.



18. To find a point that is equidistant from two given intersecting lines and at a given distance from a given point.

19. To find a point that lies in one side of a given triangle and is equidistant from the other two sides.

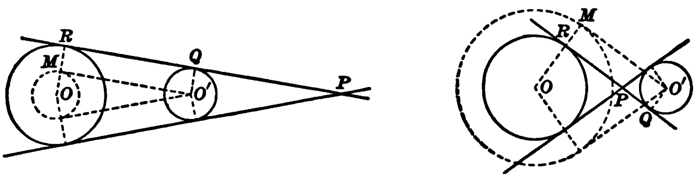
255. General Directions for solving Problems. In attacking a new problem draw the most general figure possible and the solution may be evident at once. If the solution is not evident, see if it depends on finding a point, in which case see if two loci can be found. If this is not the case, assume the problem solved and try to work backwards,—the method of analysis.

EXERCISE 39

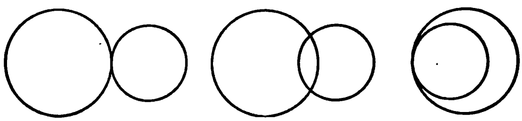
1. To draw a common tangent to two given circles.

If the centers are O and O' and the radii r and r', the tangent QR seems to be ∥ to $O'M$, a tangent from O' to a circle whose radius is $r - r'$. If this is true, we can easily reverse the process. Since there are two tangents from O', so there are two common tangents.

In the right-hand figure the tangent QR seems to be ∥ to $O'M$, a tangent from O' to a circle whose radius is $r + r'$. If this is true, we can easily reverse the process. There are four common tangents in general.



2. To draw a common tangent to two given circles, using the following figures.



3. The locus of the vertex of a right triangle, having a given hypotenuse as its base, is the circle described upon the given hypotenuse as a diameter.

4. The locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment in which the given angle may be inscribed.

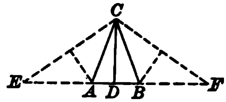
To construct an isosceles triangle, having given:

5. The base and the angle at the vertex.

6. The base and the radius of the circumscribed circle.

7. The base and the radius of the inscribed circle.

8. The perimeter and the altitude.



Let ABC be the $\triangle$ required, EF the given perimeter. The altitude CD passes through the middle of EF, and the $\triangle$ EAC, BFC are isosceles.

To construct a right triangle, having given:

9. The hypotenuse and one side.

10. One side and the altitude upon the hypotenuse.

11. The median and the altitude upon the hypotenuse.

12. The hypotenuse and the altitude upon the hypotenuse.

13. The radius of the inscribed circle and one side.

14. The radius of the inscribed circle and an acute angle.

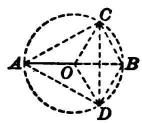
To construct a triangle, having given:

15. The base, the altitude, and an angle at the base.

16. The base, the altitude, and the angle at the vertex.

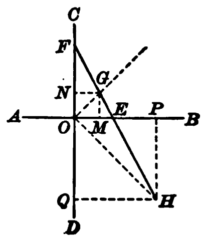
17. One side, an adjacent angle, and the sum of the other sides.

18. To construct an equilateral triangle, having given the radius of the circumscribed circle.

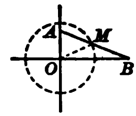


19. To construct a rectangle, having given one side and the angle between the diagonals.

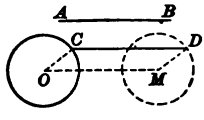
20. Given two perpendiculars, AB and CD, intersecting in O, and a line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)



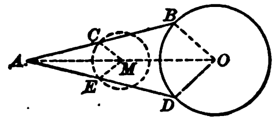
21. A straight rod moves so that its ends constantly touch two fixed rods perpendicular to each other. Find the locus of its mid-point.



22. A line moves so that it remains parallel to a given line, and so that one end lies on a given circle. Find the locus of the other end.



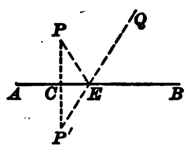
23. Find the locus of the mid-point of a line-segment that is drawn from a given external point to a given circle.



24. To draw lines from two given points P and Q which shall meet on a given line AB and make equal angles with AB.

$\because \angle BEQ = \angle PEC$, $\therefore \angle CEP' = \angle PEC$. (Why ?)

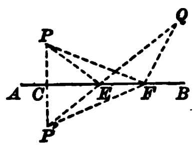
But it is easy to make $\angle CEP' = \angle PEC$, by making $PP' \perp AB$, and $CP' = PC$, and joining P' and Q.



25. To find the shortest path from a point P to a line AB and thence to a point Q.

Prove that $PE + EQ < PF + FQ$, where $\angle BEQ = \angle PEC$.

This shows that a ray of light from a point to a plane mirror and thence to another point takes the shortest possible path.



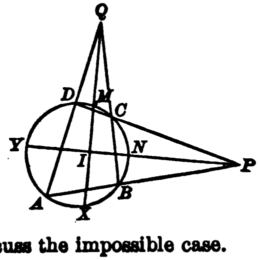
26. The bisectors of the angles included by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.

Arc AX − arc MD
= arc XB − arc CM. (Why ?)

Arc YA − arc BN
= arc DY − arc NC. (Why ?)

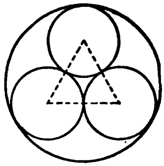
$\therefore$ arc YX + arc NM
= arc MY + arc XN. (Why ?)

$\therefore \angle YIX = \angle XIN$. (Why ?)



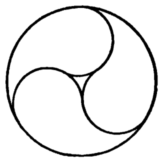
How does this prove the proposition? Discuss the impossible case.

27. Construct this design, making the figure twice this size.



Construct the equilateral $\triangle$. Then describe the small Ⓢ with half the side of the $\triangle$ as a radius. Then find the radius of the circumscribing $\odot$.

28. A circular window in a church has a design similar to the accompanying figure. Draw it, making the figure twice this size.

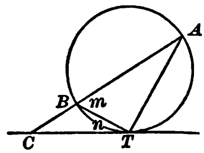


This is made from the figure of the preceding exercise, by erasing certain lines.

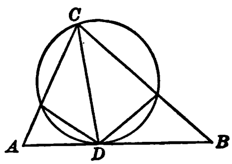
29. Two wheels of radii 1 ft. 6 in. and 2 ft. 3 in. respectively are connected by a belt, drawn straight between the points of tangency. The centers being 6 ft. apart, draw the figure mathematically. Use the scale of 1 in. to the foot.

30. A water wheel is broken and all but a fragment is lost. A workman wishes to restore the wheel. Make a drawing showing how he can construct a wheel the size of the original.

31. In this figure $\angle m = 62°$, and $\angle n = 28°$. Find the number of degrees in each of the other angles, and determine whether AB is a diameter.

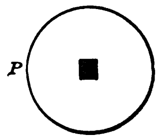


32. In this figure $\angle B = 41°$, $\angle A = 65°$, and $\angle BDC = 97°$. Find the number of degrees in each of the other angles, and determine whether CD is a diameter.



33. Construct or explain why it is impossible to construct a triangle with sides 3 in., 2 in., 6 in.; also one with sides 5 in., 7 in., 12 in.; also one with sides 2 in., 1 in., $1\frac{1}{2}$ in.

34. Show how to draw a tangent to this circle at the point P, the center of the circle not being accessible.



EXERCISE 40

1. In a circle whose center is O the chord AB is drawn so that $\angle BAO = 27°$. How many degrees are there in $\angle AOB$?

2. In a circle whose center is O the chord AB is drawn so that $\angle BAO = 25°$. On the circle, and on the same side of AB as the center O, the point D is taken and is joined to A and B. How many degrees are there in $\angle ADB$?

3. What is the locus of the mid-point of a chord of a circle formed by secants drawn from a given external point?

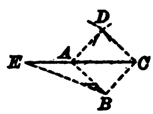
4. In a circle whose center is O two perpendiculars OM and ON are drawn to the chords AB and CD respectively, and it is known that $\angle NMO = \angle ONM$. Prove that $AB = CD$.

5. Two circles intersect at the points A and B. Through A a variable secant is drawn, cutting the circles at C and D. Prove that the angle DBC is constant.

6. Let A and B be two fixed points on a given circle, and M and N be the extremities of a rotating diameter of the same circle. Find the locus of the point of intersection of the lines AM and BN.

7. Upon a line AB a segment of a circle containing 240° is constructed, and in the segment any chord PQ subtending an arc of 60° is drawn. Find the locus of the point of intersection of AP and BQ; also of AQ and BP.

8. To construct a square, given the sum of the diagonal and one side.



Let $ABCD$ be the square required, and CA the diagonal. Produce CA, making $AE = AB$. $\triangle ABC$ and ABE are isosceles and $\angle BAC = \angle ACB = 45°$. Find the value of $\angle E$. Construct $\angle CBE$. Now reverse the reasoning.

The propositions in Exercise 40 are taken from recent college entrance examination papers.

EXERCISE 41

Review Questions

1. Define the word *circle* and the principal terms used in connection with it.

2. What is meant by a central angle? How is it measured?

3. What is meant by an inscribed angle? How is it measured?

4. State the general proposition covering all the cases that have been considered relating to the measure of an angle formed by the intersection of two secants.

5. State all of the facts you have learned relating to equal chords of a circle.

6. State all of the facts you have learned relating to unequal chords of a circle.

7. State all of the facts you have learned relating to tangents to a circle.

8. How many points are required to determine a straight line? two parallel lines? an angle? a circle?

9. Name one kind of magnitude that you have learned to trisect, and state how you proceed to trisect this magnitude.

10. In order to construct a definite triangle, what parts must be known?

11. What are the important methods of attacking a new problem in geometry? Which is the best method to try first?

12. What is meant by determinate, indeterminate, and impossible cases in the solution of a problem?

13. Distinguish between a constant and a variable, and give an illustration of each.

14. Distinguish between inscribed, circumscribed, and escribed circles.

15. What is meant by the statement that a central angle is measured by the intercepted arc?