APPENDIX

709. Subjects Treated. As with plane geometry, so with solid geometry, there are many topics that might be taken in addition to those given in any textbook. The theorems and problems already given in this work are standard propositions that are looked upon as basal, and are usually required as preliminary to more advanced work, and these, with a reasonable selection from the exercises, will be all that most schools have time to consider. It occasionally happens, however, that a school is able to do more than this, and then more exercises may be selected from the large number contained in this work, and a few additional topics may be studied. For this latter purpose the appendix is added, but its study should not be undertaken at the expense of good work on the fundamental propositions and the exercises depending upon them.

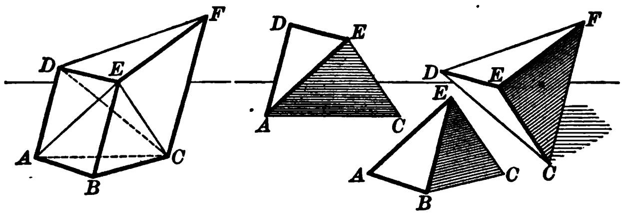
The subjects treated are certain additional propositions in the mensuration of solids, and a few general theorems relating to similar polyhedrons, these being occasionally required for college examinations. There is also added a brief sketch of the history of geometry, which all students are advised to read as a matter of general information, and a few of those recreations of geometry that add a peculiar interest to the subject.

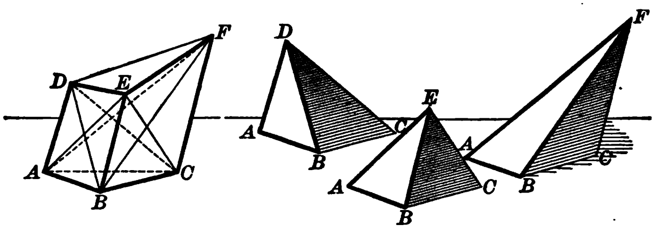
710. Similar Polyhedrons. Polyhedrons that have the same number of faces, respectively similar and similarly placed, and their corresponding polyhedral angles equal, are called *similar polyhedrons.*

It will be seen that this is analogous to the definition of similar polygons in plane geometry.

Proposition I. Theorem

711. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*





Given a truncated triangular prism *ABC-DEF* whose base is *ABC* and inclined section *DEF*, the truncated prism being divided into the three pyramids *E-ABC*, *E-ACD*, and *E-CFD*.

To prove ABC-DEF equivalent to the sum of the three pyramids E-ABC, D-ABC, and F-ABC.

Proof. *E-ABC* has the base *ABC* and the vertex *E*.

Now pyramid *E-ACD* = pyramid *B-ACD*. § 558

(*For they have the same base, ACD, and the same altitude, since their vertices E and B are in the line EB* ∥ *to the plane ACD.*)

But the pyramid *B-ACD* may be regarded as having the base *ABC* and the vertex *D*; that is, as pyramid *D-ABC*.

Then since $\triangle$ CFD and ACF have the common base CF and equal altitudes, their vertices lying in the line AD which is parallel to CF, they are equivalent. § 326

Furthermore, pyramids E-CFD and B-ACF not only have equivalent bases, the $\triangle$ CFD and ACF, but they have the same altitude, since their vertices E and B are in the line EB which is parallel to the plane of their bases.

$\therefore$ pyramid E-CFD = pyramid B-ACF. § 558

But the pyramid B-ACF may be regarded as having the base ABC and the vertex F; that is, as pyramid F-ABC.

Therefore the truncated triangular prism ABC-DEF is equivalent to the sum of the three pyramids E-ABC, D-ABC, and F-ABC. Q.E.D.

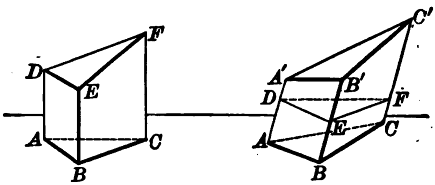


FIG. 1 FIG. 2

712. Corollary 1. *The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.*

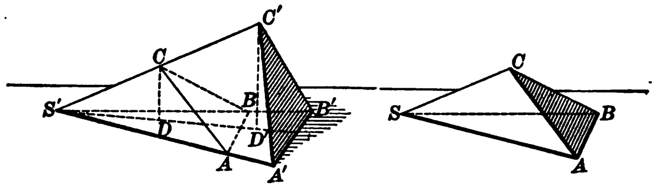
For the lateral edges DA, EB, FC (Fig. 1), being perpendicular to the base ABC, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism. It is interesting to consider the special case in which $\triangle DEF$ is parallel to $\triangle ABC$.

713. Corollary 2. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

For the right section DEF (Fig. 2) divides the truncated prism into two truncated right prisms.

Proposition II. Theorem

714. *The volumes of two tetrahedrons that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.*



Given the two tetrahedrons S-ABC and S'-$A'B'C'$, having the trihedral angles S and S' equal, v and v' denoting the volumes.

To prove that $$\frac{v}{v'} = \frac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}.$$

Proof. Place the tetrahedron S-ABC upon S'-$A'B'C'$ so that the trihedral $\angle S$ shall coincide with the equal trihedral $\angle S'$.

Draw CD and $C'D' \perp$ to the plane $S'A'B'$,

and let their plane intersect $S'A'B'$ in $S'DD'$.

The faces $S'AB$ and $S'A'B'$ may be taken as the bases, and CD, $C'D'$ as the altitudes, of the triangular pyramids C-$S'AB$ and C'-$S'A'B'$ respectively.

Then $$\frac{v}{v'} = \frac{S'AB \times CD}{S'A'B' \times C'D'} = \frac{S'AB}{S'A'B'} \times \frac{CD}{C'D'}. \qquad \S\ 562$$

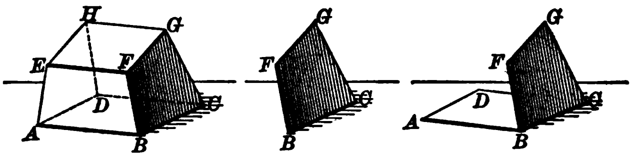
But $$\frac{S'AB}{S'A'B'} = \frac{S'A \times S'B}{S'A' \times S'B'}, \qquad \S\ 332$$

and $$\frac{CD}{C'D'} = \frac{S'C}{S'C'}. \qquad \S\ 282$$

$$\therefore \frac{v}{v'} = \frac{S'A \times S'B \times S'C}{S'A' \times S'B' \times S'C'} = \frac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'},$$ by Ax. 9. Q.E.D.

Proposition III. Theorem

715. *In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.*



Given the polyhedron AG, e denoting the number of edges, v the number of vertices, and f the number of faces.

To prove that $e + 2 = v + f$.

Proof. Beginning with one face $BCGF$, we have $e = v$.

Annex a second face $ABCD$ by applying one of its edges to a corresponding edge of the first face, and there is formed a surface of two faces having *one* edge BC and *two* vertices B and C common to the two faces.

Therefore for two faces $e = v + 1$.

Annex a third face $ABFE$, adjoining each of the first two faces. This face will have *two* edges AB, BF and *three* vertices A, B, F in common with the surface already formed.

Therefore for three faces $e = v + 2$.

In like manner, for four faces, $e = v + 3$, and so on.

Therefore for $(f-1)$ faces $e = v + (f - 2)$.

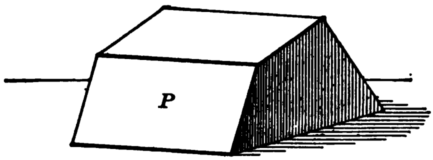
But $f-1$ is the number of faces of the polyhedron when only one face is lacking, and the addition of this face will not increase the number of edges or vertices. Hence for f faces

$$e = v + f - 2, \text{ or } e + 2 = v + f. \qquad \text{Q.E.D.}$$

This theorem is due to the great Swiss mathematician, Euler.

Proposition IV. Theorem

716. *The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.*



Given the polyhedron P, e denoting the number of edges, v the number of vertices, f the number of faces, and s the sum of the face angles.

To prove that $s = (v - 2)\, 4$ *rt.* $\angle$s.

Proof. Since e denotes the number of edges, $2e$ will denote the number of sides of the faces, considered as independent polygons, for each edge is common to two polygons.

If an exterior angle is formed at each vertex of every polygon, the sum of the interior and exterior angles at each vertex is 2 rt. $\angle$s; and since there are $2e$ vertices, the sum of the interior and exterior angles of all the faces is

$$2e \times 2 \text{ rt. } \angle s, \text{ or } e \times 4 \text{ rt. } \angle s.$$

But the sum of the ext. $\angle$s of each face is 4 rt. $\angle$s. § 146

Therefore the sum of all the ext. $\angle$s of f faces is

$$f \times 4 \text{ rt. } \angle s.$$

Therefore $s = e \times 4 \text{ rt. } \angle s - f \times 4 \text{ rt. } \angle s$

$= (e - f)\, 4 \text{ rt. } \angle s.$

But $e + 2 = v + f$; § 715

that is, $e - f = v - 2$. Ax. 2

Therefore $s = (v - 2)\, 4 \text{ rt. } \angle s$. Q.E.D.

EXERCISE 115

Find the volumes of truncated triangular prisms, given the bases b, and the distances of the three vertices p, q, r from the planes of the bases, as follows:

1. $b = 8$ sq. in., $p = 3$ in., $q = 4$ in., $r = 5$ in.
2. $b = 9$ sq. in., $p = 6$ in., $q = 3$ in., $r = 4\frac{1}{2}$ in.
3. $b = 15$ sq. in., $p = 7$ in., $q = 9$ in., $r = 8.1$ in.
4. $b = 32$ sq. in., $p = 9$ in., $q = 12$ in., $r = 9.3$ in.
5. $b = 48$ sq. in., $p = 16$ in., $q = 15$ in., $r = 18$ in.

6. A triangular rod of iron is cut square off (i.e. in right section) at one end, and slanting at the other end. The right section is an equilateral triangle $1\frac{1}{2}$ in. on a side. The edges of the rod are 3 ft. 2 in., 3 ft. 3 in., and 3 ft. 3 in. Find the weight of the rod, allowing 0.28 lb. per cubic inch.

7. Two triangular pyramids with a trihedral angle of the one equal to a trihedral angle of the other have the edges of these angles 3 in., 4 in., $3\frac{1}{2}$ in., and 5 in., $5\frac{1}{2}$ in., 6 in. respectively. Find the ratio of the volumes.

8. Make a table giving the number of edges, vertices, and faces of each of the five regular polyhedrons, showing that in every case the number conforms to Euler's theorem (§ 715).

9. Make a table similar to that of Ex. 8, giving the sum of the face angles in each of the five regular polyhedrons, showing that in every case $s = (v - 2)\ 4$ rt. $\angle s$ (§ 716).

10. There can be no seven-edged polyhedron.

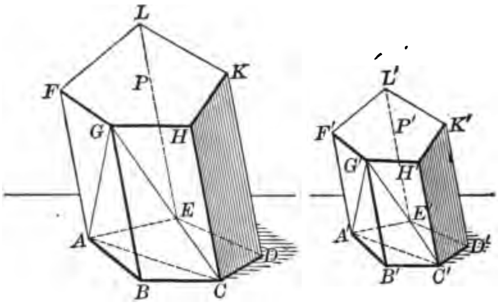
11. Can there be a nine-edged polyhedron?

12. What is the sum of the face angles of a six-edged polyhedron?

13. What is the sum of the face angles of a polyhedron with five vertices? with four vertices? Consider the possibility of a polyhedron with three vertices.

Proposition V. Theorem

717. *Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.*



Given two similar polyhedrons *P* and *P'*.

To prove that P and P' can be separated into the same number of tetrahedrons similar each to each and similarly placed.

Proof. Let G and G' be corresponding vertices.

Divide all the faces of P and P', except those which include the angles G and G', into corresponding triangles by drawing corresponding diagonals.

Pass a plane through G and each diagonal of the faces of P; also pass a plane through G' and each corresponding diagonal of P'.

Any two corresponding tetrahedrons $G\text{-}ABC$ and $G'\text{-}A'B'C'$ have the faces ABC, GAB, GBC similar respectively to the faces $A'B'C'$, $G'A'B'$, $G'B'C'$. § 292

Since $$\frac{AG}{A'G'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{GC}{G'C'},$$ § 282

$\therefore$ the face GAC is similar to the face $G'A'C'$. § 289

They also have the corresponding trihedral ∡ equal. § 498

∴ the tetrahedron $G\text{-}ABC$ is similar to $G'\text{-}A'B'C'$. § 710

If $G\text{-}ABC$ and $G'\text{-}A'B'C'$ are removed, the polyhedrons remaining continue similar; for the new faces GAC and $G'A'C'$ have just been proved similar, and the modified faces AGF and $A'G'F'$, GCH and $G'C'H'$, are similar (§ 292); also the modified polyhedral ∡ G and G', A and A', C and C' remain equal each to each, since the corresponding parts taken from these angles are equal.

The process of removing similar tetrahedrons can be carried on until the polyhedrons are separated into the same number of tetrahedrons similar each to each and similarly placed. Q.E.D.

718. Corollary 1. *The corresponding edges of similar polyhedrons are proportional.*

For the corresponding faces are similar. Therefore their corresponding sides are proportional (§ 282).

719. Corollary 2. *Any two corresponding lines in two similar polyhedrons have the same ratio as any two corresponding edges.*

For these lines may be shown to be sides of similar polygons, and hence § 282 applies.

720. Corollary 3. *Two corresponding faces of similar polyhedrons are proportional to the squares on any two corresponding edges.*

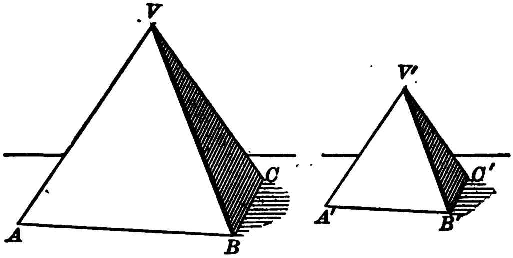
For they are similar polyhedrons, and hence they are to each other as the squares on any two corresponding sides (§ 334).

721. Corollary 4. *The entire surfaces of two similar polyhedrons are proportional to the squares on any two corresponding edges.*

For the corresponding faces are proportional to the squares on any two corresponding edges (§ 720), and hence their sum has the same proportion, by § 269.

Proposition VI. Theorem

722. *The volumes of two similar tetrahedrons are to each other as the cubes on any two corresponding edges.*



Given two similar tetrahedrons $V\text{-}ABC$ and $V'\text{-}A'B'C'$, with volumes v and v', VB and $V'B'$ being two corresponding edges.

To prove that $$\frac{v}{v'}=\frac{\overline{VB}^3}{\overline{V'B'}^3}.$$

Proof. Since the two polyhedrons are similar, Given

∴ the corresponding polyhedral angles are equal, § 710

and, in particular, the trihedral angles V and V' are equal.

$$\therefore \frac{v}{v'}=\frac{VB\times VC\times VA}{V'B'\times V'C'\times V'A'} \qquad \text{§ 714}$$

$$=\frac{VB}{V'B'}\times\frac{VC}{V'C'}\times\frac{VA}{V'A'}.$$

Furthermore, since the tetrahedrons are similar, Given

$$\therefore \frac{VB}{V'B'}=\frac{VC}{V'C'}=\frac{VA}{V'A'}. \qquad \text{§ 718}$$

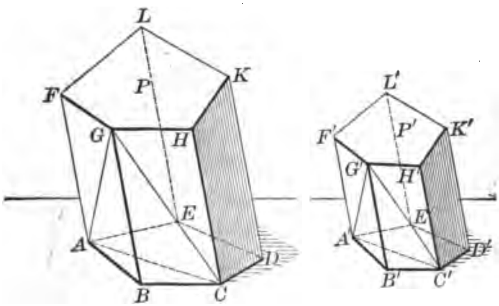
Substituting $\frac{VB}{V'B'}$ for its equals, we have

$$\frac{v}{v'}=\frac{VB}{V'B'}\times\frac{VB}{V'B'}\times\frac{VB}{V'B'}, \qquad \text{Ax. 9}$$

or $$\frac{v}{v'}=\frac{\overline{VB}^3}{\overline{V'B'}^3}. \qquad \text{Q.E.D.}$$

Proposition VII. Theorem

723. *The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.*



Given two similar polyhedrons P and P', with volumes v and v', GB and $G'B'$ being any two corresponding edges.

To prove that $v : v' = \overline{GB}^3 : \overline{G'B'}^3$.

Proof. Separate P and P' into tetrahedrons similar each to each and similarly placed (§ 717), denoting their respective volumes by $v_1, v_2, v_3, \cdots, v_1', v_2', v_3', \cdots$.

Then since $v_1 : v_1' = \overline{GB}^3 : \overline{G'B'}^3$,

$v_2 : v_2' = \overline{GB}^3 : \overline{G'B'}^3$, and so on. § 722

$\therefore v_1 + v_2 + v_3 + \cdots : v_1' + v_2' + v_3' + \cdots = \overline{GB}^3 : \overline{G'B'}^3$. § 269

But $v_1 + v_2 + v_3 + \cdots = v$, and $v_1' + v_2' + v_3' + \cdots = v'$.

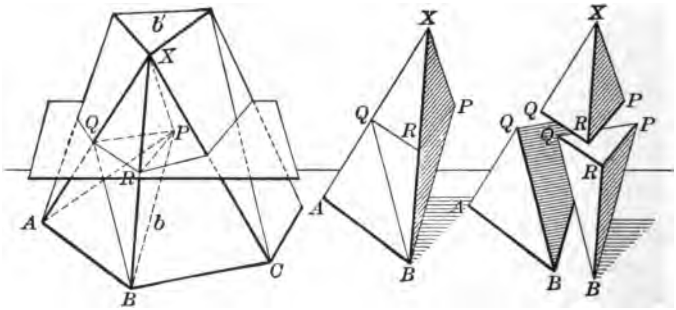
$\therefore v : v' = \overline{GB}^3 : \overline{G'B'}^3$, by Ax. 9. Q.E.D.

724. Prismatoid. A polyhedron having for bases two polygons in parallel planes, and for lateral faces triangles or trapezoids with one side common with one base, and the opposite vertex or side common with the other base, is called a *prismatoid*.

The *altitude* is the distance between the planes of the bases. The *mid-section* is the section made by a plane parallel to the bases and bisecting the altitude.

PROPOSITION VIII. THEOREM

725. *The volume of a prismatoid is equal to the product of one sixth of its altitude into the sum of its bases and four times its mid-section.*



Given a prismatoid of volume v, bases b and b', mid-section m, and altitude a.

To prove that $v = \frac{1}{6} a (b + b' + 4m)$.

Proof. If any lateral face is a trapezoid, divide it into two triangles by a diagonal.

Take any point P in the mid-section and join P to the vertices of the polyhedron and of the mid-section.

Separate the prismatoid into pyramids which have their vertices at P, and for their respective bases the lower base b, the upper base b', and the lateral faces of the prismatoid.

The pyramid P-XAB, which we may call a lateral pyramid, is composed of the three pyramids P-XQR, P-QBR, and P-QAB.

Now P-XQR may be regarded as having vertex X and base PQR, and P-QBR as having vertex B and base PQR.

Hence the volume of P-XQR is equal to $\frac{1}{6} a \cdot PQR$,

and the volume of P-QBR is equal to $\frac{1}{6} a \cdot PQR$. § 559

The pyramids P-QAB and P-QBR have the same vertex P. The base QAB is twice the base QBR (§ 327), since the $\triangle\ QAB$ has its base AB twice the base QR of the $\triangle\ QBR$ (§ 136), and these triangles have the same altitude (§ 724).

Hence the pyramid P-QAB is equivalent to twice the pyramid P-QBR. § 563

Hence the volume of P-QAB is equal to $\frac{2}{6}\,a \cdot PQR$.

Therefore the volume of P-XAB, which is composed of P-XQR, P-QBR, and P-QAB, is equal to $\frac{4}{6}\,a \cdot PQR$.

In like manner, the volume of each lateral pyramid is equal to $\frac{4}{6}\,a \times$ the area of that part of the mid-section which is included within it; and therefore the total volume of all these lateral pyramids is equal to $\frac{4}{6}\,am$.

The volume of the pyramid with base b is $\frac{1}{6}\,ab$,
and the volume of the pyramid with base b' is $\frac{1}{6}\,ab'$. § 559

Therefore $v = \frac{1}{6}\,a\,(b + b' + 4\,m)$. Q.E.D.

EXERCISE 116

Deduce from the formula for the volume of a prismatoid, $v = \frac{1}{6}\,a\,(b + b' + 4\,m)$, *the following formulas:*

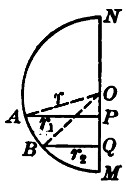
1. Cube, $v = a^3$.
2. Prism, $v = ba$.
3. Pyramid, $v = \frac{1}{3}\,ba$.
4. Parallelepiped, $v = ba$.
5. Frustum of a pyramid, $v = \frac{1}{3}\,a\left(b + b' + \sqrt{bb'}\right)$.

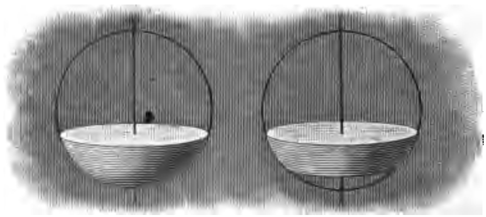
6. A prismatoid has an upper base 3 sq. in., a lower base 7 sq. in., an altitude 3 in., and a mid-section 4 sq. in. What is the volume?

7. A wedge has for its base a rectangle l in. long and w in. wide. The cutting edge is e in. long, and is parallel to the base. The distance from e to the base is d in. Deduce a formula for the volume of the wedge. Apply this formula to the case in which $l = 6$, $w = 1$, $e = 5$, $d = 3$.

Proposition IX. Theorem

726. *The volume of a spherical segment is equal to the product of one half the sum of its bases by its altitude, increased by the volume of a sphere having that altitude for its diameter.*





Given a spherical segment of volume v, generated by the revolution of $ABQP$ about MN as an axis, r being the radius of the sphere, AP being represented by r_1, BQ by r_2, and PQ by a.

To prove that $v = \frac{1}{2} a(\pi r_1^2 + \pi r_2^2) + \frac{1}{6}\pi a^3$.

Proof. We shall first find the volume of the spherical segment with one base, generated by AMP.

Area of zone $AM = 2\pi r \cdot PM$. § 691

$\therefore$ volume of sector generated by $OAM = \frac{1}{3}r \times 2\pi r \cdot PM$. § 708

But the cone generated by $OAP = \frac{1}{3}\pi r_1^2(r - PM)$. § 611

$\therefore$ volume $AMP = \frac{1}{3}r \times 2\pi r \cdot PM \cdot - \frac{1}{3}\pi r_1^2(r - PM)$. Ax. 2

But $r_1^2 = PM \times NP = PM(2r - PM)$. § 297

$\therefore$ volume $AMP = \frac{1}{3}r \times 2\pi r \cdot PM$

$- \frac{1}{3}\pi \cdot PM(2r - PM)(r - PM)$ Ax. 9

$= \pi \cdot \overline{PM}^2(r - \frac{1}{3}PM)$.

In the same way, volume $BMQ = \pi \cdot \overline{QM}^2(r - \frac{1}{3}QM)$.

$\therefore v =$ volume $AMP -$ volume BMQ

$= \pi \cdot \overline{PM}^2 \cdot r - \frac{1}{3}\pi \cdot \overline{PM}^3 - \pi \cdot \overline{QM}^2 \cdot r + \frac{1}{3}\pi \cdot \overline{QM}^3$

$= \pi r(\overline{PM}^2 - \overline{QM}^2) - \frac{1}{3}\pi(\overline{PM}^3 - \overline{QM}^3)$.

But $PM - QM = a$. Given

$\therefore v = \pi r a(PM + QM) - \frac{1}{3}\pi a(\overline{PM}^2 + PM \cdot QM + \overline{QM}^2)$. Ax. 9

But $a^2 = \overline{PM}^2 - 2\,PM \cdot QM + \overline{QM}^2$. Ax. 5

$\therefore a^2 + 3\,PM \cdot QM = \overline{PM}^2 + PM \cdot QM + \overline{QM}^2$. Ax. 1

$\therefore v = \pi r a(PM + QM) - \frac{1}{3}\pi a(a^2 + 3\,PM \cdot QM)$. Ax. 9

Furthermore $(2r - PM)PM = r_1^2$,

and $(2r - QM)QM = r_2^2$. § 297

$\therefore 2r \cdot PM + 2r \cdot QM - \overline{PM}^2 - \overline{QM}^2 = r_1^2 + r_2^2$. Ax. 1

$\therefore r \cdot PM + r \cdot QM = \dfrac{r_1^2 + r_2^2}{2} + \dfrac{\overline{PM}^2 + \overline{QM}^2}{2}$. Axs. 1, 4

$$\therefore v = \pi a\left(\frac{r_1^2 + r_2^2}{2} + \frac{\overline{PM}^2 + \overline{QM}^2}{2} - \frac{a^2}{3} - PM \cdot QM\right)$$

$$= \pi a\left(\frac{r_1^2 + r_2^2}{2} + \frac{a^2}{2} + PM \cdot QM - \frac{a^2}{3} - PM \cdot QM\right)$$

$$= \tfrac{1}{2}a(\pi r_1^2 + \pi r_2^2) + \tfrac{1}{6}\pi a^3.$$ Q.E.D.

EXERCISE 117

Find the volumes of spherical segments having bases b and b′, and altitudes a, as follows:

1. $b = 4$, $b' = 5$, $a = 1$.
2. $b = 4$, $b' = 6$, $a = 1\frac{1}{4}$.
3. $b = 5$, $b' = 7$, $a = 2\frac{1}{3}$.
4. $b = 6$, $b' = 8$, $a = 1\frac{1}{2}$.
5. $b = 8$, $b' = 12$, $a = 2$.
6. $b = 12$, $b' = 15$, $a = 3\frac{1}{2}$.
7. $b = 27$ sq. in., $b' = 32$ sq. in., $a = 2.33$ in.

Find the volumes of spherical segments having radii of bases r_1 and r_2, and altitudes a, as follows:

8. $r_1 = 3$, $r_2 = 4$, $a = 2$.
9. $r_1 = 4$, $r_2 = 7$, $a = 3$.
10. $r_1 = 8$, $r_2 = 5$, $a = 4\frac{1}{3}$.
11. $r_1 = 5$, $r_2 = 3$, $a = 1\frac{1}{3}$.
12. $r_1 = 6$, $r_2 = 5$, $a = 1\frac{1}{4}$.
13. $r_1 = 9$, $r_2 = 10$, $a = 2\frac{3}{4}$.
14. $r_1 = 9$ in., $r_2 = 7$ in., $a = 4.75$ in.

EXERCISE 118

EXAMINATION QUESTIONS

1. A pyramid 6 ft. high is cut by a plane parallel to the base, the area of the section being $\frac{1}{3}$ that of the base. How far from the vertex is the cutting plane?

2. Find the area of a spherical triangle whose angles are 100°, 120°, and 140°, the diameter of the sphere being 16 in.

3. Two angles of a spherical triangle are 80° and 120°. Find the limits of the third angle, and prove that the greatest possible area of the triangle is four times the least possible area, the sphere on which it is drawn being given.

4. An irregular portion, less than half, of a material sphere is given. Show how the radius can be found, compasses and ruler being allowed.

5. Find the volume of a cone of revolution, the area of the total surface of which is 200π sq. ft., and the altitude of which is 16 ft.

6. The volumes of two similar polyhedrons are 64 cu. ft. and 216 cu. ft. respectively. If the area of the surface of the first polyhedron is 112 sq. ft., find the area of the surface of the second polyhedron.

7. A solid sphere of metal of radius 12 in. is recast into a hollow sphere. If the cavity is spherical, of the same radius as the original sphere, find the thickness of the shell.

8. The stone spire of a church is a regular pyramid 50 ft. high on a hexagonal base each side of which is 10 ft. There is a hollow part which is also a regular pyramid 45 ft. high, on a hexagonal base of which each side is 9 ft. Find the number of cubic feet of stone in the spire.

9. The volumes of a hemisphere, right circular cone, and right circular cylinder are equal. Their bases are also equal, each being a circle of radius 10 in. Find the altitude of each.

10. A sphere of radius 5 ft. and a right circular cone also of radius 5 ft. stand on a plane. If the height of the cone is equal to a diameter of the sphere, find the position of the plane that cuts the two solids in equal circular sections.

11. The vertices of one regular tetrahedron are at the centers of the faces of another regular tetrahedron. Find the ratio of the volumes.

12. Find the area of a spherical triangle, if the perimeter of its polar triangle is 297° and the radius of the sphere is 10 centimeters.

13. The radii of two spheres are 13 in. and 15 in. respectively, and the distance between the centers is 14 in. Find the volume of the solid common to both spheres,— a spherical lens.

14. The radius of the base of a right circular cylinder is r and the altitude of the cylinder is a. Find the radius and the volume of a sphere whose surface is equivalent to the lateral surface of the cylinder.

15. If the polyhedral angle at the vertex of a triangular pyramid is trirectangular, and the areas of the lateral faces are a, b, and c respectively, and the area of the base is d, then $a^2+b^2+c^2=d^2$.

16. If the earth is a sphere with a diameter of 8000 mi., find the area of the zone bounded by the parallels 30° north latitude and 30° south latitude. Show that this zone and the planes of the circles include $\frac{11}{16}$ of the volume of the earth.

17. The altitude of a cone of revolution is 12 centimeters and the radius of its base is 5 centimeters. Compute the radius of the sector of paper which, when rolled up, will just cover the convex surface of the cone, and compute the size of the central angle of this sector in degrees, minutes, and seconds.

18. The volume of any regular pyramid is equal to one third of its lateral area multiplied by the perpendicular distance from the center of its base to any lateral face.

19. If the area of a zone of one base is n times the area of the circle which forms its base, the altitude of the zone is $\frac{1}{n}(n-1)$ times the diameter of the sphere. Discuss the special case when $n=1$.

20. If the four sides of a spherical quadrilateral are equal, its diagonals are perpendicular to each other.

21. Find the volume of a pyramid whose base contains 30 square centimeters if one lateral edge is 5 centimeters and the angle formed by this edge and the plane of the base is 45°.

22. On the base of a right circular cone a hemisphere is constructed outside the cone. The surface of the hemisphere equals the surface of the cone. If r is the radius of the hemisphere, find the slant height of the cone, the inclination of the slant height to the base, and the volume of the entire solid.

23. Find the total surface and the volume of a regular tetrahedron whose edge equals 8 centimeters.

24. If a spherical quadrilateral is inscribed in a small circle, the sum of two opposite angles is equal to the sum of the other two angles.

25. By what number must the dimensions of a cylinder of revolution be multiplied to obtain a similar cylinder of revolution with surface n times that of the first? with volume n times that of the first?

26. A pyramid is cut by a plane parallel to the base midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.

27. The height of a regular hexagonal pyramid is 36 ft. and one side of the base is 6 ft. What are the dimensions of a similar pyramid whose volume is $\frac{1}{27}$ that of the first?

28. One of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?