

## VIII. PROGRESSIONS AND PROPORTION.

### ARITHMETICAL PROGRESSION.

**220.** A **Series** is a succession of quantities or numbers, connected together by the signs  $+$  or  $-$ , in which succeeding terms may be derived from those which precede them, by a rule deducible from the *law* of the series.

Thus,  $1+3+5+7+9+$ , etc.,  
 $2+6+18+54+$ , etc., are series.

In the former, any term may be derived from that which precedes it, by adding 2; and, in the latter, any term may be found by multiplying the preceding term by 3.

**221.** An **Arithmetical Progression** is a series of quantities which increase or decrease, by a *common difference*.

Thus, the numbers 1, 3, 5, 7, 9, etc., form an *increasing* arithmetical progression, in which the common difference is 2.

The numbers 30, 27, 24, 21, 18, 15, etc., form a *decreasing* arithmetical progression, in which the common difference is 3.

**REMARK.**—An arithmetical progression is termed, by some writers, an *equidifferent series*, or a *progression by difference*.

Again,  $a, a+d, a+2d, a+3d, a+4d, a+5d$ , etc., is an *increasing* arithmetical progression, whose first term is  $a$ , and common difference  $d$ .

If  $d$  be negative, it becomes  $a, a-d, a-2d, a-3d, a-4d, a-5d$ , etc., which is a *decreasing* arithmetical progression.

**222.** If we take an arithmetical series, of which the first term is  $a$ , and common difference  $d$ , we have

$$\begin{aligned} \text{1st term} &= \dots \dots \dots a; \\ \text{2d term} &= \text{1st term} + d = a + d; \\ \text{3d term} &= \text{2d term} + d = a + 2d; \\ \text{4th term} &= \text{3d term} + d = a + 3d; \text{ and so on.} \end{aligned}$$

Hence, the coefficient of  $d$  in any term is less by unity, than the number of that term in the series; therefore, the  $n$ th term  $= a + (n-1)d$ .

If we designate the  $n$ th term by  $l$ , we have  $l = a + (n-1)d$ .

For a decreasing series we also have  $l = a - (n-1)d$ .  
Hence,

#### TO FIND ANY TERM OF AN ARITHMETICAL SERIES,

**Rule.—1. FOR AN INCREASING SERIES.**—*Multiply the common difference by the number of terms less one, and add the product to the first term.*

**2. FOR A DECREASING SERIES.**—*Multiply the common difference by the number of terms less one, and subtract the product from the first term.*

1. The first term of an increasing arithmetical series is 3, and common difference 5; required the 8th term.

$$\text{Here } l, \text{ or 8th term, } = 3 + (8-1)5 = 3 + 35 = 38, \text{ Ans.}$$

2. The first term of a decreasing arithmetical series is 50, and common difference 3; required the 10th term.

$$\text{Here } l, \text{ or 10th term, } = 50 - (10-1)3 = 50 - 27 = 23, \text{ Ans.}$$

**REVIEW.**—220. What is a series? 221. What is an arithmetical progression? Give an example of an increasing series. Of a decreasing series.

222. Rule for finding the last term of an increasing arithmetical series. Of a decreasing series. Prove these rules.

In the following examples,  $a$  denotes the first term, and  $d$  the common difference of an arithmetical series;  $d$  being *plus* when the series is *increasing*, and *minus* when it is *decreasing*.

3.  $a=3$ , and  $d=5$ ; required the 6th term. Ans. 28.

4.  $a=7$ , and  $d=\frac{1}{4}$ ; required the 16th term. Ans.  $10\frac{3}{4}$ .

5.  $a=2\frac{1}{2}$ , and  $d=\frac{1}{3}$ ; required the 100th term. Ans.  $35\frac{1}{2}$ .

6.  $a=0$ , and  $d=\frac{1}{2}$ ; required the 11th term. Ans. 5.

7.  $a=30$ , and  $d=-2$ ; required the 8th term. Ans. 16.

8.  $a=-10$ , and  $d=-2$ ; required the 6th term.

Ans. -20.

9. If a body falls during 20 sec., descending  $16\frac{1}{12}$  ft. the first sec.,  $48\frac{1}{4}$  ft. the next, and so on, how far will it fall the twentieth sec.? Ans.  $627\frac{1}{4}$  ft.

**223.** Given, the first term  $a$ , the common difference  $d$ , and the number of terms  $n$ , to find  $s$ , the sum of the series.

If we take an arithmetical series, of which the first term is 3, common difference 2, and number of terms 5, it may be written in the following forms:

$$\begin{array}{ccccc} 3, & 5, & 7, & 9, & 11, \\ 11, & 9, & 7, & 5, & 3. \end{array}$$

It is obvious that the sum of all the terms in either of these lines will represent the sum of the series; that is,

$$\begin{array}{lcl} & s = 3 + 5 + 7 + 9 + 11 & \\ \text{And} & s = 11 + 9 + 7 + 5 + 3 & \\ \text{Adding,} & 2s = 14 + 14 + 14 + 14 + 14 & \\ & = 14 \times 5, \text{ the number of terms, } = 70. & \\ \text{Whence,} & s = \frac{1}{2} \text{ of } 70 = 35. & \end{array}$$

Now, let  $l$  = the last term, and  $n$  = the number of terms. Writing the series as before,

$$\begin{array}{lcl} & s = a + (a+d) + (a+2d) + (a+3d) + \dots + l & \\ \text{And} & s = l + (l-d) + (l-2d) + (l-3d) + \dots + a & \\ \text{Adding,} & 2s = (l+a) + (l+a) + (l+a) + (l+a) + \dots + (l+a) & \\ \text{Hence,} & 2s = (l+a)n, \text{ and} & \\ & s = (l+a) \frac{n}{2} = \left( \frac{l+a}{2} \right) n. \text{ Hence,} & \end{array}$$

## TO FIND THE SUM OF AN ARITHMETICAL SERIES,

**Rule.**—*Multiply half the sum of the two extremes by the number of terms.*

From the preceding, it appears that the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.

Since  $l = a + (n-1)d$ , if we substitute this in the place of  $l$  in the formula  $s = (l + a) \frac{n}{2}$ , it becomes  $s = (2a + (n-1)d) \frac{n}{2}$ . Hence,

## TO FIND THE SUM OF AN ARITHMETICAL SERIES,

**Rule.**—*To twice the first term, add the product of the number of terms less one, by the common difference, and multiply the sum by half the number of terms.*

1. Find the sum of an arithmetical series, of which the first term is 3, last term 17, and number of terms 8.

$$s = \left( \frac{3+17}{2} \right) 8 = 80, \text{ Ans.}$$

2. Find the sum of an arithmetical series, whose first term is 1, last term 12, and number of terms 12.

Ans. 78.

3. Find the sum of an arithmetical series, whose first term is 0, common difference 1, and number of terms 20.

Ans. 190.

4. Find the sum of an arithmetical series, whose first term is 3, common difference 2, and number of terms 21.

Ans. 483.

**REVIEW.**—223. What is the rule for finding the sum of an arithmetical series? Prove the rule.

5. Find the sum of an arithmetical series, whose first term is 10, common difference  $-3$ , and number of terms 10.

Ans.  $-35$ .

**224.** The equations,  $l=a+(n-1)d$

$$s=(a+l)\frac{n}{2},$$

furnish the means of solving this general problem:

*Knowing any three of the five quantities  $a$ ,  $d$ ,  $n$ ,  $l$ ,  $s$ , which enter into an arithmetical series, to determine the other two.*

This question furnishes ten cases, for the solution of which we have always two equations, with only two unknown quantities.

1. Let it be required to find  $a$  in terms of  $l$ ,  $n$ ,  $d$ .

From the first formula, by transposing, we have  $a=l-(n-1)d$ ; that is,

*The first term of an increasing arithmetical series is equal to the last term diminished by the product of the common difference into the number of terms less one.*

From the same formula, we find  $d=\frac{l-a}{n-1}$ ; that is,

*In any arithmetical series, the common difference is equal to the difference of the extremes, divided by the number of terms less one.*

**225.** By means of the preceding rules, we are enabled to solve such problems as the following:

REVIEW.—224. What are the fundamental equations of arithmetical progression, and to what general problem do they give rise?

224. To what is the first term of an increasing arithmetical series equal? To what is the common difference of an arithmetical series equal?

Let it be required to insert five arithmetical means between 3 and 15.

Here, the two given terms with the five to be inserted make seven. Hence,  $n=7$ ,  $a=3$ ,  $l=15$ , from which we find  $d=2$ . Adding the common difference to 3 and the succeeding terms, we have for the series 3, 5, 7, 9, 11, 13, 15.

If we insert the same number of means between the consecutive terms of a series, the result will form a new progression. Thus,

If we insert 3 terms between the terms in 1, 9, 17, etc., the new series will be 1, 3, 5, 7, 9, 11, 13, 15, 17, etc.

1. Find the sum of the natural series of numbers 1, 2, 3, 4, . . . . carried to 1000 terms.    Ans. 500500.

2. Required the last term, and the sum of the series, 1, 3, 5, 7, . . . . to 101 terms.    Ans. 201 and 10201.

3. How many times does a common clock strike in a week?    Ans. 1092.

4. Find the  $n$ th term, and the sum of  $n$  terms of the natural series of numbers 1, 2, 3, 4, . . . .  
Ans.  $n$ , and  $\frac{1}{2}n(n+1)$ .

5. The first and last terms of an arithmetical series are 2 and 29, and the common difference is 3; required the number of terms and the sum of the series.  
Ans. 10 and 155.

6. The first and last terms of a decreasing arithmetical series are 10 and 6, and the number of terms 9; required the common difference, and the sum of the series.  
Ans.  $\frac{1}{2}$  and 72.

7. The first term of a decreasing arithmetical series is 10, the number of terms 10, and the sum of the series 85; required the last term and the common difference.  
Ans. 7 and  $\frac{1}{3}$ .

8. Required the series obtained from inserting four arithmetical means between each of the two terms of the series 1, 16, 31, etc.    Ans. 1, 4, 7, 10, 13, 16, etc.

9. The sum of an arithmetical progression is 72, the first term is 24, and the common difference is  $-4$ ; required the number of terms. Ans. 9, or 4.

This question presents the equation  $n^2 - 13n = -36$ , which has two roots, 9 and 4. These give rise to the two following series, in each of which the sum is 72.

First series, 24, 20, 16, 12, 8, 4, 0,  $-4$ ,  $-8$ .

Second series, 24, 20, 16, 12.

10. A man bought a farm, paying for the first A. \$1, for the second \$2, for the third \$3, and so on; when he came to settle, he had to pay \$12880; how many A. did the farm contain, and what was the average price per A.?

Ans. 160 A., at  $\$80\frac{1}{2}$  per A.

11. If A start from a certain place, traveling  $a$  mi. the first da.,  $2a$  the second,  $3a$  the third, and so on; and at the end of 4 da., B start after him from the same place, traveling uniformly  $9a$  mi. a da.; when will B overtake A?

Let  $x$  = the number of da. required; then, the distance traveled by A in  $x$  da.  $= a + 2a + 3a$ , etc., to  $x$  terms,  $= \frac{1}{2}ax(x+1)$ ; and the distance traveled by B in  $(x-4)$  da.  $= 9a(x-4)$ .

Whence,  $\frac{1}{2}ax(x+1) = 9a(x-4)$ . From which  $x=8$ , or 9.

Hence, B overtakes A at the end of 8 da.; and since, on the ninth da., A travels  $9a$  mi., which is B's uniform rate, they will be together at the end of the ninth da.

12. A sets out 3 hr. and 20 min. before B, and travels at the rate of 6 mi. an hr.; in how many hr. will B overtake A, if he travel 5 mi. the first hr., 6 the second, 7 the third, and so on? Ans. 8 hr.

13. A and B set out from the same place, at the same time. A travels at the constant rate of 3 mi. an hr., but B's rate of traveling is 4 mi. the first hr.,  $3\frac{1}{2}$  the second, 3 the third, and so on; in how many hr. will A overtake B? Ans. 5 hr.

REVIEW.—225. How do you insert any number of arithmetical means between two given numbers?

## GEOMETRICAL PROGRESSION.

**226.** A Geometrical Progression is a series of terms, each of which is derived from the preceding, by multiplying it by a constant quantity, termed the *ratio*.

Thus, 1, 2, 4, 8, 16, etc., is an *increasing* geometrical series, whose common ratio is 2.

Also, 54, 18, 6, 2, etc., is a *decreasing* geometrical series, whose common ratio is  $\frac{1}{3}$ .

Generally,  $a, ar, ar^2, ar^3$ , etc., is a geometrical progression, whose common ratio is  $r$ , and which is an *increasing* or *decreasing* series, according as  $r$  is *greater* or *less* than 1.

It is obvious that the common ratio will be ascertained by dividing any term of the series by that which precedes it.

**227.** To find the last term of the series.

Let  $a$  denote the 1st term,  $r$  the ratio,  $l$  the  $n$ th term, and  $s$  the sum of  $n$  terms; then, the respective terms of the series will be

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5 & . & . & . & n-3, & n-2, & n-1, & n \\ a, & ar, & ar^2, & ar^3, & ar^4 & . & . & . & ar^{n-4}, & ar^{n-3}, & ar^{n-2}, & ar^{n-1}. \end{array}$$

That is, the exponent of  $r$  in the *second* term is 1, in the *third* 2, in the *fourth* 3, and so on; the  $n$ th term of the series will be,  $l = ar^{n-1}$ . Hence,

## TO FIND ANY TERM OF A GEOMETRICAL SERIES,

**Rule.**—Multiply the first term by the ratio raised to a power, whose exponent is one less than the number of terms.

1. Find the 5th term of the geometric progression, whose first term is 4, and common ratio 3.

$$l = 4 \times 3^4 = 4 \times 81 = 324, \text{ the fifth term.}$$



2. Find the 6th term of the progression 2, 8, 32, etc.  
Ans. 2048.
3. Given the 1st term 1, and ratio 2, to find the 7th term.  
Ans. 64.
4. Given the 1st term 4, and ratio 3, to find the 10th term.  
Ans. 78732.
5. Find the 9th term of the series, 2, 10, 50, etc.  
Ans. 781250.
6. Given the first term 8, and ratio  $\frac{1}{2}$ , to find the 15th term.  
Ans.  $20\frac{1}{4}$ .
7. A man purchased 9 horses, agreeing to pay for the whole what the last would cost, at \$2 for the first, \$6 for the second, etc.; what was the average price of each?  
Ans. \$1458.

**228.** To find the sum of all the terms of the series.

Let  $a, ar, ar^2, ar^3$ , etc., be any geometrical series, and  $s$  its sum; then,

$$s = a + ar + ar^2 + ar^3 \dots + ar^{n-2} + ar^{n-1}$$

Multiplying this equation by  $r$ , we have

$$rs = ar + ar^2 + ar^3 + ar^4 \dots + ar^{n-1} + ar^n.$$

The terms of the two series are identical, except the *first* term of the first series, and the *last* term of the second series. Subtracting the first equation from the second, we have

$$rs - s = ar^n - a$$

Or,  $(r-1)s = a(r^n - 1)$

Hence,  $s = \frac{a(r^n - 1)}{r - 1}$

Since  $l = ar^{n-1}$ , we have  $rl = ar^n$

Therefore,  $s = \frac{ar^n - a}{r - 1} = \frac{rl - a}{r - 1}$ . Hence,

**REVIEW.**—226. What is a geometrical progression? Give example of an increasing geometrical series. Of a decreasing. How may the common ratio in any geometrical series be found?

227. How is any term of a geometrical series found? Explain the principle of this rule.

## TO FIND THE SUM OF A GEOMETRICAL SERIES,

**Rule.**—Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.

1. Find the sum of 10 terms of the progression 2, 6, 18, 54, etc.

$$\text{The last term} = 2 \times 3^9 = 2 \times 19683 = 39366.$$

$$s = \frac{lr - a}{r - 1} = \frac{118098 - 2}{3 - 1} = 59048, \text{ Ans.}$$

2. Find the sum of 7 terms of the progression 1, 2, 4, etc. Ans. 127.

3. Find the sum of 10 terms of the progression 4, 12, 36, etc. Ans. 118096.

4. Find the sum of 8 terms of the series, whose first term is  $6\frac{1}{4}$ , and ratio  $\frac{3}{2}$ . Ans.  $307\frac{441}{12}$ .

5. Find the sum of  $3 + 4\frac{1}{2} + 6\frac{3}{4} +$ , etc., to 5 terms. Ans.  $39\frac{9}{16}$ .

If the ratio  $r$  is less than 1, the progression is decreasing, and the last term  $lr$  is less than  $a$ . To render both terms of the fraction positive, change the signs of the terms, Art. 132, and we have  $s = \frac{a - rl}{1 - r}$ , for the sum of the series when the progression is decreasing.

6. Find the sum of 15 terms of the series 8, 4, 2, 1, etc. Ans.  $15\frac{2047}{2048}$ .

7. Find the sum of 6 terms of the series 6,  $4\frac{1}{2}$ ,  $3\frac{3}{8}$ , etc. Ans.  $19\frac{373}{128}$ .

**REVIEW.**—228. Rule for the sum of the terms of a geometrical series. Prove this rule. When the series is decreasing, how may the formula be written so that both terms of the fraction may be positive?

**229.** In a decreasing geometrical series, when the number of terms is infinite, the last term becomes *infinitely small*, that is, 0. Therefore,  $rl=0$ , and the formula

$$s = \frac{a - rl}{1 - r} \text{ becomes } s = \frac{a}{1 - r}. \text{ Hence,}$$

TO FIND THE SUM OF AN INFINITE DECREASING SERIES,

**Rule.**—*Divide the first term by one minus the ratio.*

1. Find the sum of the infinite series  $1 + \frac{1}{3} + \frac{1}{9} +$ , etc.

Here,  $a=1$ ,  $r=\frac{1}{3}$ , and  $s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$ , Ans.

2. Find the sum of the infinite series  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ , etc.

Ans. 2.

3. Find the sum of the infinite series  $9 + 6 + 4 +$ , etc.

Ans. 27.

4. Find the sum of the infinite series  $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} +$ , etc.

Ans.  $\frac{3}{4}$ .

5. Find the sum of the infinite series  $1 + \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} +$ , etc.

Ans.  $\frac{x^2}{x^2-1}$ .

6. Find the sum of the infinite geometrical progression  $a - b + \frac{b^2}{a} - \frac{b^3}{a^2} + \frac{b^4}{a^3} -$ , etc., in which the ratio is  $-\frac{b}{a}$ .

Ans.  $\frac{a^2}{a+b}$ .

7. A body moves 10 ft. the first sec., 5 the next,  $2\frac{1}{2}$  the next, and so on forever; how many ft. would it move over?

Ans. 20.

**REVIEW.**—229. Rule for the sum of a decreasing geometrical series, when the number of terms is infinite. Prove this rule.

**230.** The equations,  $l = ar^{n-1}$ , and  $s = \frac{ar^n - a}{r - 1}$  furnish this general problem :

*Knowing three of the five quantities  $a$ ,  $r$ ,  $n$ ,  $l$ , and  $s$ , of a geometrical progression, to find the other two.*

This problem embraces ten different questions, as in arithmetical progression. Some of these, however, involve the extraction of high roots, the application of logarithms, and the solution of higher equations than those treated of in the preceding pages.

The following is one of the most simple and useful of these cases :

Having given the first and last terms, and the number of terms of a geometrical progression, to find the ratio.

$$\text{Here, } l = ar^{n-1}, \text{ or } r^{n-1} = \frac{l}{a}. \quad \text{Hence, } r = \sqrt[n-1]{\left(\frac{l}{a}\right)}.$$

1. The first and last terms of a geometrical series are 3 and 48, the number of terms 5 ; required the intermediate terms.

$$\begin{aligned} \text{Here, } l &= 48, a = 3, n - 1 = 5 - 1 = 4. \\ \text{Hence, } r &= \sqrt[4]{\frac{48}{3}} = \sqrt[4]{16} = 2. \end{aligned}$$

2. In a geometrical series of three terms, the first and last terms are 4 and 16 ; required the middle term.

Ans. 8.

In a geometrical progression of three terms, the middle term is called a *mean proportional* between the other two.

3. Find a mean proportional between 8 and 32.

Ans. 16.

4. The first and last terms of a geometrical series are 2 and 162, and the number of terms 5 ; required the ratio.

Ans. 3.