UNIT 12 Quadratic Equations

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List of Objectives

To solve a quadratic equation by factoring

To solve a quadratic equation by taking square roots

To solve a quadratic equation by completing the square

To solve a quadratic equation by using the quadratic formula

To graph a quadratic equation of the form $y = ax^2 + bx + c$

To solve application problems

SECTION 1 Solving Quadratic Equations by Factoring or by Taking Square Roots

1.1 Objective To solve a quadratic equation by factoring

An equation of the form $ax^2 + bx + c = 0$, where a, b, and c are constants and $a > 0$, is a **quadratic equation.**

$4x^2 - 3x + 1 = 0$, $a = 4$, $b = -3$, $c = 1$

$3x^2 - 4 = 0$, $a = 3$, $b = 0$, $c = -4$

A quadratic equation is also called a **second-degree equation.**

A quadratic equation is in **standard form** when the polynomial is in descending order and equal to zero.

Recall that the Principle of Zero Products states that if the product of two factors is zero, then at least one of the factors must be zero.

If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

The Principle of Zero Products can be used in solving quadratic equations.

Solve by factoring: $2x^2 - x = 1$

$$2x^2 - x = 1$$

Write the equation in standard form. $2x^2 - x - 1 = 0$

Factor. $(2x + 1)(x - 1) = 0$

Let each factor equal zero. $2x + 1 = 0 \qquad x - 1 = 0$

Rewrite each equation in the form *variable = constant*. $2x = -1 \qquad x = 1$

$$x = -\frac{1}{2}$$

Write the solutions. The solutions are $-\frac{1}{2}$ and 1.

Check:

$$2x^2 - x = 1 \qquad 2x^2 - x = 1$$
$$2\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) = 1 \qquad 2(1)^2 - 1 = 1$$
$$2 \cdot \frac{1}{4} + \frac{1}{2} = 1 \qquad 2 \cdot 1 - 1 = 1$$
$$\frac{1}{2} + \frac{1}{2} = 1 \qquad 2 - 1 = 1$$
$$1 = 1 \qquad 1 = 1$$

Example 1 Solve by factoring: $x^2 + 10x + 25 = 0$

Solution $x^2 + 10x + 25 = 0$

$(x + 5)(x + 5) = 0$

$x + 5 = 0 \qquad x + 5 = 0$

$x = -5 \qquad x = -5$

-5 is a double root of the equation.

The solution is -5.

Example 2 Solve by factoring: $2x^2 = (x + 2)(x + 3)$

Your solution

Solution on p. 497

1.2 Objective To solve a quadratic equation by taking square roots

The quadratic equation $x^2 = 25$ can be read "the square of a number equals 25." The solution is the positive or the negative square root of 25, 5 or -5.

$x^2 = 25$	$x^2 = 25$
$5^2 \mid 25$	$(-5)^2 \mid 25$
$25 = 25$	$25 = 25$
5 is a solution.	-5 is a solution.

The solution can be found by taking the square root of each side of the equation and writing the positive and the negative square roots of the number.
$x = \pm 5$ means $x = 5$ or $x = -5$.

$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = \pm\sqrt{25} = \pm 5$$

The solutions are 5 and -5.

Solve by taking square roots: $3x^2 = 36$

$$3x^2 = 36$$

Solve for x^2.

$$x^2 = 12$$

Take the square root of each side of the equation.

$$\sqrt{x^2} = \sqrt{12}$$

Simplify.

$$x = \pm\sqrt{12} = \pm 2\sqrt{3}$$

Write the solutions.

The solutions are $2\sqrt{3}$ and $-2\sqrt{3}$.

Check:

$3x^2 = 36$	$3x^2 = 36$
$3(2\sqrt{3})^2 = 36$	$3(-2\sqrt{3})^2 = 36$
$3(12) = 36$	$3(12) = 36$
$36 = 36$	$36 = 36$

An equation containing the square of a binomial can be solved by taking square roots.

Solve by taking square roots: $2(x-1)^2 - 36 = 0$

$$2(x-1)^2 - 36 = 0$$

Solve for $(x-1)^2$.

$$2(x-1)^2 = 36$$
$$(x-1)^2 = 18$$

Take the square root of each side of the equation.

$$\sqrt{(x-1)^2} = \sqrt{18}$$

Simplify.

$$x - 1 = \pm\sqrt{18} = \pm 3\sqrt{2}$$

Solve for x.

$x - 1 = 3\sqrt{2}$	$x - 1 = -3\sqrt{2}$
$x = 1 + 3\sqrt{2}$	$x = 1 - 3\sqrt{2}$

Write the solutions.

The solutions are $1 + 3\sqrt{2}$ and $1 - 3\sqrt{2}$.

$1 + 3\sqrt{2}$ and $1 - 3\sqrt{2}$ check as solutions.

Example 3 Solve by taking square roots: $x^2 + 16 = 0$

Solution

$$x^2 + 16 = 0$$
$$x^2 = -16$$
$$\sqrt{x^2} = \sqrt{-16}$$

$\sqrt{-16}$ is not a real number.

The equation has no real number solution.

Example 4 Solve by taking square roots: $x^2 + 81 = 0$

Your solution

Solution on p. 497

1.1 Exercises

Solve by factoring.

1. $x^2 + 2x - 15 = 0$
2. $t^2 + 3t - 10 = 0$
3. $z^2 - 4z + 3 = 0$
4. $s^2 - 5s + 4 = 0$
5. $p^2 + 3p + 2 = 0$
6. $v^2 + 6v + 5 = 0$
7. $x^2 - 6x + 9 = 0$
8. $y^2 - 8y + 16 = 0$
9. $12y^2 + 8y = 0$
10. $6x^2 - 9x = 0$
11. $r^2 - 10 = 3r$
12. $t^2 - 12 = 4t$
13. $3v^2 - 5v + 2 = 0$
14. $2p^2 - 3p - 2 = 0$
15. $3s^2 + 8s = 3$
16. $3x^2 + 5x = 12$
17. $9z^2 = 12z - 4$
18. $6r^2 = 12 - r$
19. $4t^2 = 4t + 3$
20. $5y^2 + 11y = 12$
21. $4v^2 - 4v + 1 = 0$
22. $9s^2 - 6s + 1 = 0$
23. $x^2 - 9 = 0$
24. $t^2 - 16 = 0$
25. $4y^2 - 1 = 0$
26. $9z^2 - 4 = 0$
27. $x + 15 = x(x - 1)$
28. $p + 18 = p(p - 2)$
29. $r^2 - r - 2 = (2r - 1)(r - 3)$
30. $s^2 + 5s - 4 = (2s + 1)(s - 4)$
31. $x^2 + x + 5 = (3x + 2)(x - 4)$

1.2 Exercises

Solve by taking square roots.

32. $x^2 = 36$

33. $y^2 = 49$

34. $v^2 - 1 = 0$

35. $z^2 - 64 = 0$

36. $4x^2 - 49 = 0$

37. $9w^2 - 64 = 0$

38. $9y^2 = 4$

39. $4z^2 = 25$

40. $16v^2 - 9 = 0$

41. $25x^2 - 64 = 0$

42. $y^2 + 81 = 0$

43. $z^2 + 49 = 0$

44. $w^2 - 24 = 0$

45. $v^2 - 48 = 0$

46. $(x - 1)^2 = 36$

47. $(y + 2)^2 = 49$

48. $2(x + 5)^2 = 8$

49. $4(z - 3)^2 = 100$

50. $9(x - 1)^2 - 16 = 0$

51. $4(y + 3)^2 - 81 = 0$

52. $49(v + 1)^2 - 25 = 0$

53. $81(y - 2)^2 - 64 = 0$

54. $(x - 4)^2 - 20 = 0$

55. $(y + 5)^2 - 50 = 0$

56. $(x + 1)^2 + 36 = 0$

57. $(y - 7)^2 + 49 = 0$

58. $2\left(z - \frac{1}{2}\right)^2 = 12$

59. $3\left(v + \frac{3}{4}\right)^2 = 36$

SECTION 2 Solving Quadratic Equations by Completing the Square

2.1 Objective To solve a quadratic equation by completing the square

Recall that a perfect square trinomial is the square of a binomial.

Perfect Square Trinomial		Square of a Binomial
$x^2 + 6x + 9$	=	$(x + 3)^2$
$x^2 - 10x + 25$	=	$(x - 5)^2$
$x^2 + 8x + 16$	=	$(x + 4)^2$

For each perfect square trinomial, the square of $\frac{1}{2}$ of the coefficient of x equals the constant term.

$x^2 + 6x + 9$, $\left(\frac{1}{2} \cdot 6\right)^2 = 9$

$x^2 - 10x + 25$, $\left[\frac{1}{2}(-10)\right]^2 = 25$

$x^2 + 8x + 16$, $\left(\frac{1}{2} \cdot 8\right)^2 = 16$

$$\left(\frac{1}{2}\ \textbf{coefficient of}\ x\right)^2 = \textbf{constant term}$$

This relationship can be used to write the constant term for a perfect square trinomial. Adding to a binomial the constant term which makes it a perfect square trinomial is called **completing the square.**

Complete the square on $x^2 - 8x$. Write the resulting perfect square trinomial as the square of a binomial.

Find the constant term. $\left[\frac{1}{2}(-8)\right]^2 = 16$

Complete the square on $x^2 - 8x$ by adding the constant term. $x^2 - 8x + 16$

Write the resulting perfect square trinomial as the square of a binomial. $x^2 - 8x + 16 = (x - 4)^2$

Complete the square on $y^2 + 5y$. Write the resulting perfect square trinomial as the square of a binomial.

Find the constant term. $\left(\frac{1}{2} \cdot 5\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

Complete the square on $y^2 + 5y$ by adding the constant term. $y^2 + 5y + \frac{25}{4}$

Write the resulting perfect square trinomial as the square of a binomial. $y^2 + 5y + \frac{25}{4} = \left(y + \frac{5}{2}\right)^2$

A quadratic equation which cannot be solved by factoring can be solved by completing the square. Add to both sides of the equation the term which completes the square. Rewrite the quadratic equation in the form $(x + a)^2 = b$. Take the square root of each side of the equation and then solve for x.

Solve by completing the square: $x^2 - 6x - 3 = 0$

$$x^2 - 6x - 3 = 0$$

Add the opposite of the constant term to each side of the equation.

$$x^2 - 6x = 3$$

Find the constant term which completes the square on $x^2 - 6x$.

$$\left[\frac{1}{2}(-6)\right]^2 = 9$$

Do this step mentally.

Add this term to each side of the equation.

$$x^2 - 6x + 9 = 3 + 9$$

Factor the perfect square trinomial.

$$(x - 3)^2 = 12$$

Take the square root of each side of the equation.

$$\sqrt{(x - 3)^2} = \sqrt{12}$$

Simplify.

$$x - 3 = \pm\sqrt{12} = \pm 2\sqrt{3}$$

Solve for x.

$$x - 3 = 2\sqrt{3} \qquad x - 3 = -2\sqrt{3}$$
$$x = 3 + 2\sqrt{3} \qquad x = 3 - 2\sqrt{3}$$

Write the solution.

The solutions are $3 + 2\sqrt{3}$ and $3 - 2\sqrt{3}$.

Check:

$$x^2 - 6x - 3 = 0$$
$$(3 + 2\sqrt{3})^2 - 6(3 + 2\sqrt{3}) - 3 = 0$$
$$9 + 12\sqrt{3} + 12 - 18 - 12\sqrt{3} - 3 = 0$$
$$0 = 0$$

$$x^2 - 6x - 3 = 0$$
$$(3 - 2\sqrt{3})^2 - 6(3 - 2\sqrt{3}) - 3 = 0$$
$$9 - 12\sqrt{3} + 12 - 18 + 12\sqrt{3} - 3 = 0$$
$$0 = 0$$

Solve by completing the square: $2x^2 - x - 1 = 0$

$$2x^2 - x - 1 = 0$$

Add the opposite of the constant term to each side of the equation.

$$2x^2 - x = 1$$

To complete the square, the coefficient of the x^2 term must be 1. Multiply each term by the reciprocal of the coefficient of x^2.

$$\frac{1}{2}(2x^2 - x) = \frac{1}{2} \cdot 1$$

$$x^2 - \frac{1}{2}x = \frac{1}{2}$$

Find the term which completes the square on $x^2 - \frac{1}{2}x$.

$$\left[\frac{1}{2}\left(-\frac{1}{2}\right)\right]^2 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}$$

Do this step mentally.

Add this term to each side of the equation.

$$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{1}{2} + \frac{1}{16}$$

Factor the perfect square trinomial.

$$\left(x - \frac{1}{4}\right)^2 = \frac{9}{16}$$

Take the square root of each side of the equation.

$$\sqrt{\left(x - \frac{1}{4}\right)^2} = \sqrt{\frac{9}{16}}$$

Simplify.

$$x - \frac{1}{4} = \pm\frac{3}{4}$$

Solve for x.

$$x - \frac{1}{4} = \frac{3}{4} \qquad x - \frac{1}{4} = -\frac{3}{4}$$

$$x = 1 \qquad x = -\frac{1}{2}$$

The solutions are 1 and $-\frac{1}{2}$.

1 and $-\frac{1}{2}$ check as solutions.

Example 1 Solve by completing the square: $2x^2 - 4x - 1 = 0$

Solution

$$2x^2 - 4x - 1 = 0$$

$$2x^2 - 4x = 1$$

$$\frac{1}{2}(2x^2 - 4x) = \frac{1}{2} \cdot 1$$

$$x^2 - 2x = \frac{1}{2}$$

Complete the square.

$$x^2 - 2x + 1 = \frac{1}{2} + 1$$

$$(x - 1)^2 = \frac{3}{2}$$

$$\sqrt{(x - 1)^2} = \sqrt{\frac{3}{2}}$$

$$x - 1 = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$$

$$x - 1 = \frac{\sqrt{6}}{2} \qquad x - 1 = -\frac{\sqrt{6}}{2}$$

$$x = 1 + \frac{\sqrt{6}}{2} \qquad x = 1 - \frac{\sqrt{6}}{2}$$

$$= \frac{2 + \sqrt{6}}{2} \qquad = \frac{2 - \sqrt{6}}{2}$$

The solutions are $\frac{2 + \sqrt{6}}{2}$ and $\frac{2 - \sqrt{6}}{2}$.

Example 2 Solve by completing the square: $3x^2 - 6x - 2 = 0$

Your solution

Solution on p. 498

Example 3 Solve by completing the square:
$x^2 + 4x + 5 = 0$

Solution
$x^2 + 4x + 5 = 0$
$x^2 + 4x = -5$

Complete the square.

$x^2 + 4x + 4 = -5 + 4$
$(x + 2)^2 = -1$
$\sqrt{(x + 2)^2} = \sqrt{-1}$

$\sqrt{-1}$ is not a real number.

The quadratic equation has no real number solution.

Example 4 Solve by completing the square:
$x^2 + 6x + 12 = 0$

Your solution

Example 5 Solve by completing the square:
$x^2 + 6x + 4 = 0$
Approximate the solutions. Use the Table of Square Roots on page 438.

Solution
$x^2 + 6x + 4 = 0$
$x^2 + 6x = -4$

Complete the square.

$x^2 + 6x + 9 = -4 + 9$
$(x + 3)^2 = 5$
$\sqrt{(x + 3)^2} = \sqrt{5}$
$x + 3 = \pm\sqrt{5}$

$x + 3 = \sqrt{5}$	$x + 3 = -\sqrt{5}$
$x = -3 + \sqrt{5}$	$x = -3 - \sqrt{5}$
$\approx -3 + 2.236$	$\approx -3 - 2.236$
≈ -0.764	≈ -5.236

The solutions are approximately -0.764 and -5.236.

Example 6 Solve by completing the square:
$x^2 + 8x + 8 = 0$
Approximate the solutions. Use the Table of Square Roots on page 438.

Your solution

Solutions on p. 498

2.1 Exercises

Solve by completing the square.

1. $x^2 + 2x - 3 = 0$
2. $y^2 + 4y - 5 = 0$
3. $z^2 - 6z - 16 = 0$
4. $w^2 + 8w - 9 = 0$
5. $x^2 = 4x - 4$
6. $z^2 = 8z - 16$
7. $v^2 - 6v + 13 = 0$
8. $x^2 + 4x + 13 = 0$
9. $y^2 + 5y + 4 = 0$
10. $v^2 - 5v - 6 = 0$
11. $w^2 + 7w = 8$
12. $y^2 + 5y = -4$
13. $v^2 + 4v + 1 = 0$
14. $y^2 - 2y - 5 = 0$
15. $x^2 + 6x = 5$
16. $w^2 - 8w = 3$
17. $z^2 = 2z + 1$
18. $y^2 = 10y - 20$
19. $p^2 + 3p = 1$
20. $r^2 + 5r = 2$
21. $t^2 - 3t = -2$
22. $z^2 - 5z = -3$
23. $v^2 + v - 3 = 0$
24. $x^2 - x = 1$
25. $y^2 = 7 - 10y$
26. $v^2 = 14 + 16v$
27. $r^2 - 3r = 5$
28. $s^2 + 3s = -1$
29. $t^2 - t = 4$
30. $y^2 + y - 4 = 0$
31. $x^2 - 3x + 5 = 0$
32. $z^2 + 5z + 7 = 0$
33. $2t^2 - 3t + 1 = 0$
34. $2x^2 - 7x + 3 = 0$
35. $2r^2 + 5r = 3$
36. $2y^2 - 3y = 9$

Solve by completing the square.

37. $2s^2 = 7s - 6$

38. $2x^2 = 3x + 20$

39. $2v^2 = v + 1$

40. $2z^2 = z + 3$

41. $3r^2 + 5r = 2$

42. $3t^2 - 8t = 3$

43. $3y^2 + 8y + 4 = 0$

44. $3z^2 - 10z - 8 = 0$

45. $4x^2 + 4x - 3 = 0$

46. $4v^2 + 4v - 15 = 0$

47. $6s^2 + 7s = 3$

48. $6z^2 = z + 2$

49. $6p^2 = 5p + 4$

50. $6t^2 = t - 2$

51. $4v^2 - 4v - 1 = 0$

52. $2s^2 - 4s - 1 = 0$

53. $4z^2 - 8z = 1$

54. $3r^2 - 2r = 2$

55. $3y - 6 = (y - 1)(y - 2)$

56. $7s + 55 = (s + 5)(s + 4)$

57. $4p + 2 = (p - 1)(p + 3)$

58. $v - 10 = (v + 3)(v - 4)$

Solve by completing the square. Approximate the solutions to the nearest thousandth. Use the Table of Square Roots on page 438.

59. $y^2 + 3y = 5$

60. $w^2 + 5w = 2$

61. $2z^2 - 3z = 7$

62. $2x^2 + 3x = 11$

63. $4x^2 + 6x - 1 = 0$

64. $4x^2 + 2x - 3 = 0$

SECTION 3 Solving Quadratic Equations by Using the Quadratic Formula

3.1 Objective To solve a quadratic equation by using the quadratic formula

Any quadratic equation can be solved by completing the square. Applying this method to the standard form of a quadratic equation produces a formula that can be used to solve any quadratic equation.

Solve $ax^2 + bx + c = 0$ by completing the square.

$$ax^2 + bx + c = 0$$

Add the opposite of the constant term to each side of the equation.

$$ax^2 + bx + c + (-c) = 0 + (-c)$$

$$ax^2 + bx = -c$$

Multiply each side of the equation by the reciprocal of a, the coefficient of x^2.

$$\frac{1}{a}(ax^2 + bx) = \frac{1}{a}(-c)$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Complete the square by adding $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ to each side of the equation.

$$x^2 + \frac{b}{a}x + \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 - \frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Simplify the right side of the equation.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \left(\frac{c}{a}\cdot\frac{4a}{4a}\right)$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Factor the perfect square trinomial on the left side of the equation.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of each side of the equation.

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\left(x + \frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Solve for x.

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The Quadratic Formula

The solution of $ax^2 + bx + c = 0$ is $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The quadratic formula is frequently written in the form $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

Solve by using the quadratic formula: $2x^2 = 4x - 1$

Write the equation in standard form.

$a = 2$, $b = -4$, and $c = 1$.

$$2x^2 = 4x - 1$$
$$2x^2 - 4x + 1 = 0$$

Replace a, b, and c in the quadratic formula by their values.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}$$

Simplify.

$$= \frac{4 \pm \sqrt{16 - 8}}{4} = \frac{4 \pm \sqrt{8}}{4}$$
$$= \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2}$$

Write the solutions.

The solutions are $\frac{2 + \sqrt{2}}{2}$ and $\frac{2 - \sqrt{2}}{2}$.

$\frac{2 + \sqrt{2}}{2}$ and $\frac{2 - \sqrt{2}}{2}$ check as solutions.

Example 1 Solve by using the quadratic formula: $2x^2 - 3x + 1 = 0$

Solution $2x^2 - 3x + 1 = 0$

$a = 2$, $b = -3$, $c = 1$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2 \cdot 2}$$
$$= \frac{3 \pm \sqrt{9 - 8}}{4} = \frac{3 \pm \sqrt{1}}{4} = \frac{3 \pm 1}{4}$$

$$x = \frac{3 + 1}{4} = \frac{4}{4} = 1 \qquad x = \frac{3 - 1}{4} = \frac{2}{4} = \frac{1}{2}$$

The solutions are 1 and $\frac{1}{2}$.

Example 2 Solve by using the quadratic formula: $3x^2 + 4x - 4 = 0$

Your solution

Example 3 Solve by using the quadratic formula: $2x^2 = 8x - 5$

Solution

$$2x^2 = 8x - 5$$
$$2x^2 - 8x + 5 = 0$$

$a = 2$, $b = -8$, $c = 5$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(5)}}{2 \cdot 2}$$
$$= \frac{8 \pm \sqrt{64 - 40}}{4} = \frac{8 \pm \sqrt{24}}{4}$$
$$= \frac{8 \pm 2\sqrt{6}}{4} = \frac{4 \pm \sqrt{6}}{2}$$

The solutions are $\frac{4 + \sqrt{6}}{2}$ and $\frac{4 - \sqrt{6}}{2}$.

Example 4 Solve by using the quadratic formula: $x^2 + 2x = 1$

Your solution

Solutions on p. 499

3.1 Exercises

Solve by using the quadratic formula.

1. $x^2 - 4x - 5 = 0$
2. $y^2 + 3y + 2 = 0$
3. $z^2 - 2z - 15 = 0$
4. $v^2 + 5v + 4 = 0$
5. $z^2 + 6z - 7 = 0$
6. $s^2 + 3s - 10 = 0$
7. $t^2 + t - 6 = 0$
8. $x^2 - x - 2 = 0$
9. $y^2 = 2y + 3$
10. $w^2 = 3w + 18$
11. $r^2 = 5 - 4r$
12. $z^2 = 3 - 2z$
13. $2y^2 - y - 1 = 0$
14. $2t^2 - 5t + 3 = 0$
15. $w^2 + 3w + 5 = 0$
16. $x^2 - 2x + 6 = 0$
17. $p^2 - p = 0$
18. $2v^2 + v = 0$
19. $4t^2 - 9 = 0$
20. $4s^2 - 25 = 0$
21. $4y^2 + 4y = 15$
22. $4r^2 + 4r = 3$
23. $3t^2 = 7t + 6$
24. $3x^2 = 10x + 8$
25. $5z^2 + 11z = 12$
26. $4v^2 = v + 3$
27. $6s^2 - s - 2 = 0$
28. $6y^2 + 5y - 4 = 0$
29. $2x^2 + x + 1 = 0$
30. $3r^2 - r + 2 = 0$
31. $t^2 - 2t = 5$
32. $y^2 - 4y = 6$
33. $t^2 + 6t - 1 = 0$

Solve by using the quadratic formula.

34. $z^2 + 4z + 1 = 0$

35. $w^2 = 4w + 9$

36. $y^2 = 8y + 3$

37. $4t^2 - 4t - 1 = 0$

38. $4x^2 - 8x - 1 = 0$

39. $v^2 + 6v + 1 = 0$

40. $s^2 + 4s - 8 = 0$

41. $4t^2 - 12t - 15 = 0$

42. $4w^2 - 20w + 5 = 0$

43. $9y^2 + 6y - 1 = 0$

44. $9s^2 - 6s - 2 = 0$

45. $4p^2 + 4p + 1 = 0$

46. $9z^2 + 12z + 4 = 0$

47. $2x^2 = 4x - 5$

48. $3r^2 = 5r - 6$

49. $4p^2 + 16p = -11$

50. $4y^2 - 12y = -1$

51. $4x^2 = 4x + 11$

52. $4s^2 + 12s = 3$

53. $9v^2 = -30v - 23$

54. $9t^2 = 30t + 17$

Solve by using the quadratic formula. Approximate the solutions to the nearest thousandth. Use the Table of Square Roots on page 438.

55. $x^2 - 2x - 21 = 0$

56. $y^2 + 4y - 11 = 0$

57. $s^2 - 6s - 13 = 0$

58. $w^2 + 8w - 15 = 0$

59. $2p^2 - 7p - 10 = 0$

60. $3t^2 - 8t - 1 = 0$

61. $4z^2 + 8z - 1 = 0$

62. $4x^2 + 7x + 1 = 0$

63. $5v^2 - v - 5 = 0$

SECTION 4 Graphing Quadratic Equations in Two Variables

4.1 Objective To graph a quadratic equation of the form $y = ax^2 + bx + c$

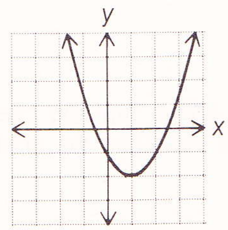
An equation of the form $y = ax^2 + bx + c$ is a **quadratic equation in two variables.** Examples of quadratic equations in two variables are shown at the right.

$y = 3x^2 - x + 1$

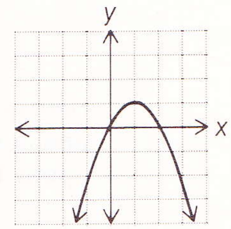
$y = -x^2 - 3$

$y = 2x^2 - 5x$

The graph of a quadratic equation in two variables is a **parabola.** The graph is "cup shaped" and opens either up or down. The graphs of two parabolas are shown below.



Parabola which opens up



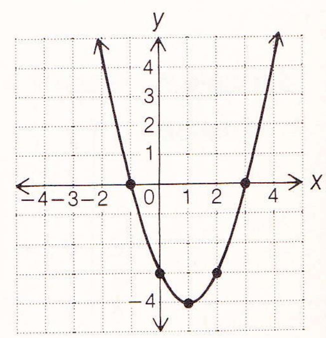
Parabola which opens down

Graph $y = x^2 - 2x - 3$.

Find several solutions of the equation. Since the graph is not a straight line, several solutions must be found in order to determine the cup shape.
Display the ordered pair solutions in a table.

x	y
0	−3
1	−4
−1	0
2	−3
3	0

Graph the ordered pair solutions on a rectangular coordinate system.
Draw a parabola through the points.

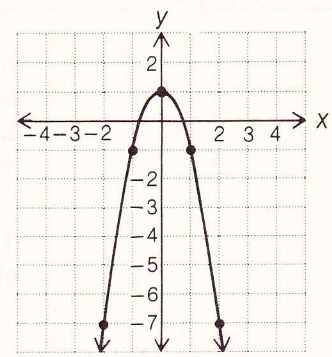


Graph $y = -2x^2 + 1$.

Find enough solutions of the equation to determine the cup shape.
Display the ordered pair solutions in a table.

x	y
0	1
1	−1
−1	−1
2	−7
−2	−7

Graph the ordered pair solutions on a rectangular coordinate system.
Draw a parabola through the points.

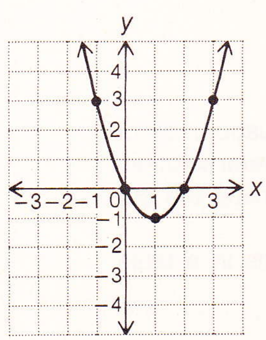


Note in the first example above that the coefficient of x^2 is **positive** and the graph **opens up.** In the second example, the coefficient of x^2 is **negative** and the graph **opens down.**

Example 1 Graph $y = x^2 - 2x$.

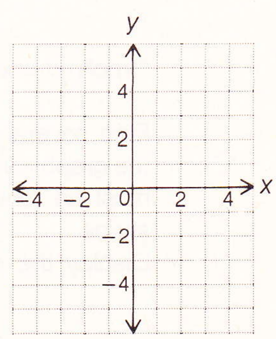
Solution

x	y
0	0
1	−1
−1	3
2	0
3	3



Example 2 Graph $y = x^2 + 2$.

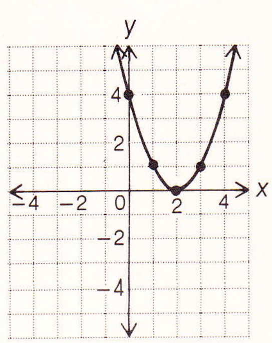
Your solution



Example 3 Graph $y = x^2 - 4x + 4$.

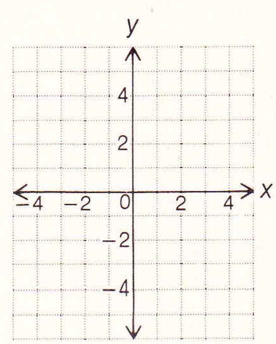
Solution

x	y
0	4
1	1
2	0
3	1
4	4



Example 4 Graph $y = x^2 + 2x + 1$.

Your solution

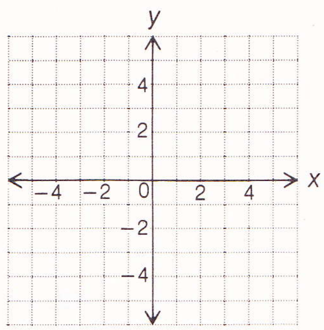


Solutions on p. 499

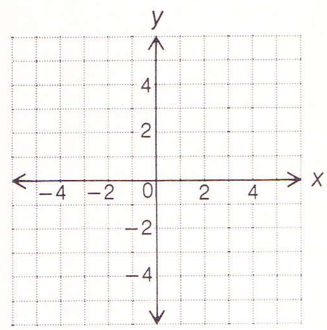
4.1 Exercises

Graph.

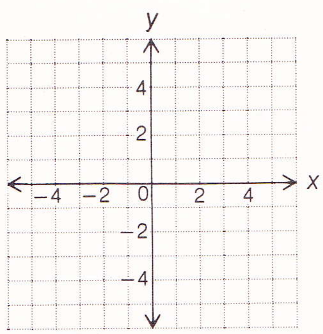
1. $y = x^2$



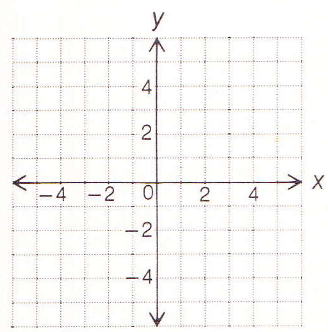
2. $y = -x^2$



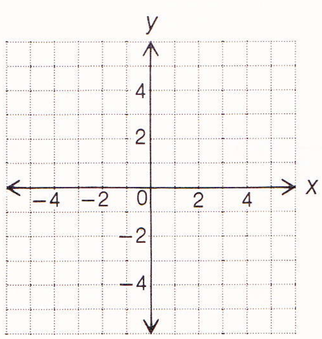
3. $y = -x^2 + 1$



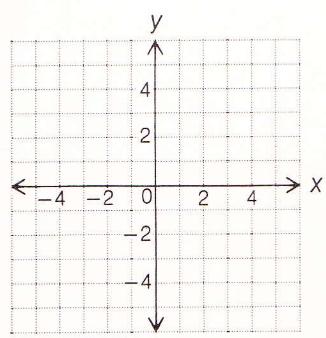
4. $y = x^2 - 1$



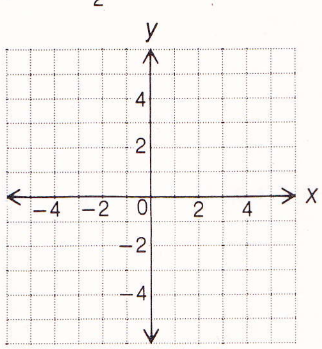
5. $y = 2x^2$



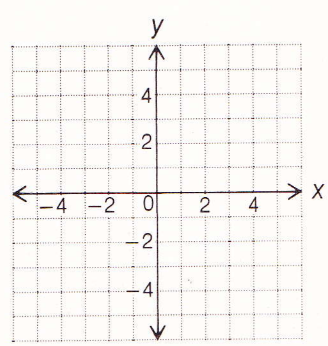
6. $y = \frac{1}{2}x^2$



7. $y = -\frac{1}{2}x^2 + 1$

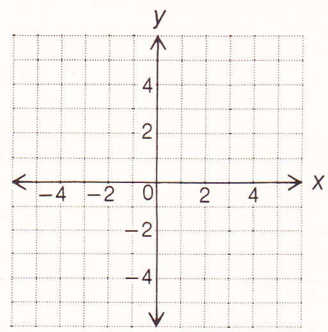


8. $y = 2x^2 - 1$

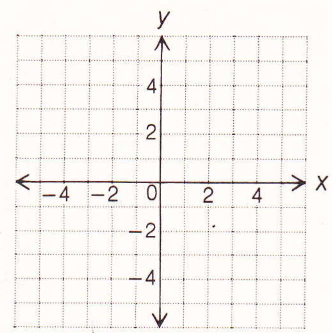


Graph.

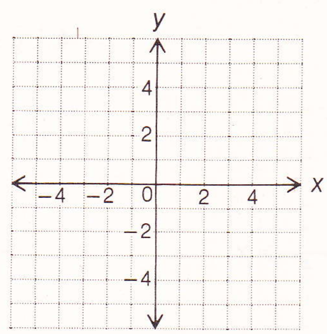
9. $y = x^2 - 4x$



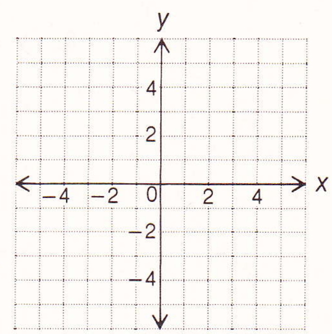
10. $y = x^2 + 4x$



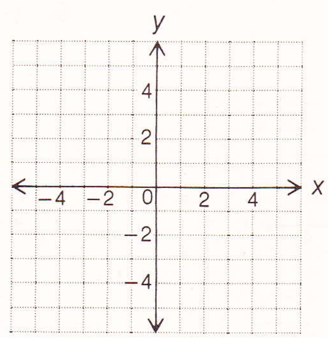
11. $y = x^2 - 2x + 5$



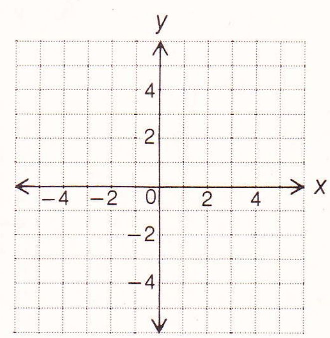
12. $y = x^2 - 4x + 2$



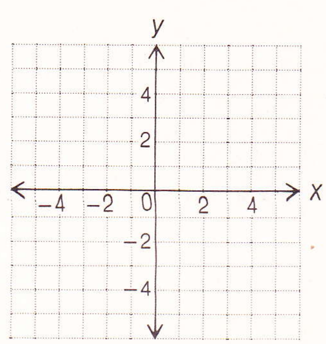
13. $y = -x^2 + 2x + 3$



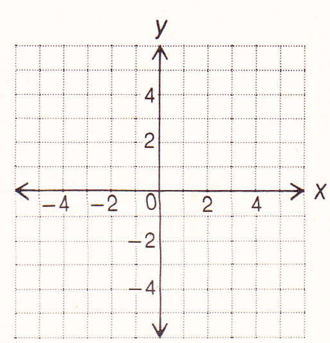
14. $y = -x^2 - 2x + 3$



15. $y = -x^2 + 4x - 4$



16. $y = -x^2 + 6x - 9$



SECTION 5 Application Problems

5.1 Objective To solve application problems

The application problems in this section are varieties of those problems solved earlier in the text. Each of the strategies for the problems in this section will result in a quadratic equation.

In 5 h, two campers rowed 12 mi down a stream and then rowed back to their campsite. The rate of the stream's current was 1 mph. Find the rate at which the campers rowed.

STRATEGY FOR SOLVING AN APPLICATION PROBLEM

▷ Determine the type of problem. For example, is it a distance-rate problem, a geometry problem, a work problem, or an age problem.

The problem is a distance-rate problem.

▷ Choose a variable to represent the unknown quantity. Write numerical or variable expressions for all the remaining quantities. These results can be recorded in a table.

The unknown rate of the campers: r

	Distance	÷	Rate	=	Time
Downstream	12	÷	$r+1$	=	$\frac{12}{r+1}$
Upstream	12	÷	$r-1$	=	$\frac{12}{r-1}$

▷ Determine how the quantities are related. If necessary, review the strategies presented in Unit 4.

The total time of the trip was 5 h.

$$\frac{12}{r+1} + \frac{12}{r-1} = 5$$

$$(r+1)(r-1)\left(\frac{12}{r+1} + \frac{12}{r-1}\right) = (r+1)(r-1)5$$

$$(r-1)12 + (r+1)12 = (r^2-1)5$$

$$12r - 12 + 12r + 12 = 5r^2 - 5$$

$$24r = 5r^2 - 5$$

$$0 = 5r^2 - 24r - 5$$

$$0 = (5r+1)(r-5)$$

$$5r + 1 = 0 \qquad r - 5 = 0$$

$$5r = -1 \qquad r = 5$$

$$r = -\frac{1}{5}$$

The solution $r = -\frac{1}{5}$ is not possible, since the rate cannot be a negative number.

The rowing rate was 5 mph.

Example 1

A painter and the painter's apprentice working together can paint a room in 2 h. The apprentice working alone requires 3 more hours to paint the room than the painter requires working alone. How long does it take the painter working alone to paint the room?

Strategy

▷ This is a work problem.

▷ Time for the painter to paint the room: t
Time for the apprentice to paint the room: $t + 3$

	Rate	Time	Part
Painter	$\frac{1}{t}$	2	$\frac{2}{t}$
Apprentice	$\frac{1}{t+3}$	2	$\frac{2}{t+3}$

▷ The sum of the parts of the task completed must equal 1.

Solution

$$\frac{2}{t} + \frac{2}{t+3} = 1$$

$$t(t+3)\left(\frac{2}{t} + \frac{2}{t+3}\right) = t(t+3) \cdot 1$$

$$(t+3)2 + t(2) = t(t+3)$$

$$2t + 6 + 2t = t^2 + 3t$$

$$0 = t^2 - t - 6$$

$$0 = (t-3)(t+2)$$

$t - 3 = 0 \qquad t + 2 = 0$

$t = 3 \qquad t = -2$

The solution $t = -2$ is not possible.

The time is 3 h.

Example 2

The length of a rectangle is 2 m more than the width. The area is 15 m^2. Find the width.

Your strategy

Your solution

Solution on p. 500

5.1 Application Problems

Solve.

1. The length of a rectangle is twice the width. The area of the rectangle is 32 ft^2. Find the length and width. (Area $= l \cdot w$)

2. The height of a triangle is four times the length of the base. The area of the triangle is 18 m^2. Find the height and the length of the base of the triangle. $\left(\text{Area} = \frac{1}{2}bh\right)$

3. The height of a triangle is 2 m more than twice the length of the base. The area of the triangle is 20 m^2. Find the height and the length of the base of the triangle.

4. The length of a rectangle is 1 ft more than twice the width. The area of the rectangle is 120 ft^2. Find the length and width of the rectangle.

5. The sum of the squares of two consecutive positive odd integers is thirty-four. Find the two integers.

6. The difference between the squares of two consecutive positive even integers is twenty-eight. Find the two integers.

7. The sum of the squares of three consecutive integers is two. Find the three integers.

8. The sum of the squares of three consecutive even integers is eight. Find the three integers.

9. An integer plus twice the square of the integer is 21. Find the integer.

10. Twice the sum of three times an integer and the square of the integer is 36. Find the integer.

11. One car is two years older than a second car. Two years ago the product of their ages was 24. Find the present ages of the two cars.

12. One coin is twice the age of a second coin. One year ago the product of their ages was 10. Find the present ages of the coins.

Solve.

13. One stamp is three times the age of a second stamp. Eight years ago the product of their ages was 19. Find the present ages of the stamps.

14. One child is twice the age of a second child. Three years ago the product of the sum of their ages and the difference between their ages was 45. Find the present ages of the children.

15. A small pipe takes 8 h longer to fill a tank than a larger pipe. Working together, the pipes can fill the tank in 3 h. How long would it take each pipe working alone to fill the tank?

16. One painter takes 6 h longer to paint a room than does a second painter. Working together, the painters can paint the room in 4 h. How long would it take each painter working alone to paint the room?

17. One photocopy machine takes 16 min longer to reproduce a report than does a second machine. Working together, it takes 6 min to reproduce the report. How long would it take each machine working alone to reproduce the report?

18. A water tank has two drains. One drain takes 21 min longer to empty the tank than does the second drain. With both drains open, the tank empties in 10 min. How long would it take each drain working alone to empty the tank?

19. A motorboat traveled 24 mi at a constant rate before reducing the speed by 2 mph. Another 20 mi was traveled at the reduced speed. The total time for the 44-mile trip was 4 h. Find the rate of the boat during the first 24 mi.

20. A motorist traveled 120 mi at a constant rate before increasing the speed by 10 mph. Another 100 mi was driven at the increased speed. The total time for the 220-mile trip was 5 h. Find the rate during the first 120 mi.

21. It took a motorboat one more hour to travel 48 mi against the current than it did to go 48 mi with the current. The rate of the current was 2 mph. Find the rate of the boat in calm water.

22. It took a small plane one more hour to fly 240 mi against the wind than it did to fly the same distance with the wind. The rate of the wind was 20 mph. Find the rate of the plane in calm air.

Review/Test

SECTION 1

1.1 Solve by factoring: $3x^2 + 7x = 20$

1.2 Solve by taking square roots: $3(x + 4)^2 - 60 = 0$

SECTION 2

2.1a Solve by completing the square: $x^2 + 4x - 16 = 0$

2.1b Solve by completing the square: $x^2 + 3x = 8$

2.1c Solve by completing the square: $2x^2 - 6x + 1 = 0$

2.1d Solve by completing the square: $2x^2 + 8x = 3$

SECTION 3

3.1a Solve by using the quadratic formula: $x^2 + 4x + 2 = 0$

3.1b Solve by using the quadratic formula: $x^2 - 3x = 6$

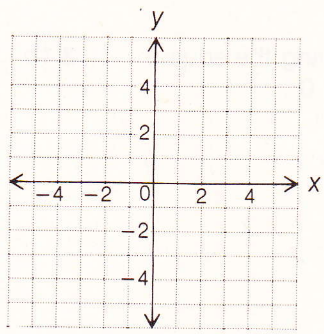
Review/Test

3.1c Solve by using the quadratic formula: $2x^2 - 5x - 3 = 0$

3.1d Solve by using the quadratic formula: $3x^2 - x = 1$

SECTION **4**

4.1 Graph $y = x^2 + 2x - 4$.



SECTION **5**

5.1a The length of a rectangle is 2 ft less than twice the width. The area of the rectangle is 40 ft^2. Find the length and width of the rectangle.

5.1b It took a motorboat one hour more to travel 60 mi against a current than it did to go 60 mi with the current. The rate of the current was 1 mph. Find the rate of the boat in calm water.

Review/Test

SECTION 1

1.1 Solve by factoring: $6x^2 - 17x = -5$

a) $\frac{2}{3}$ and 3

b) $\frac{5}{2}$ and $\frac{1}{3}$

c) $-\frac{5}{2}$ and $-\frac{1}{3}$

d) $-\frac{2}{3}$ and -3

1.2 Solve by taking square roots: $2(x - 5)^2 = 36$

a) 6 and -6

b) $3\sqrt{2}$ and $-3\sqrt{2}$

c) $5 + 3\sqrt{2}$ and $5 - 3\sqrt{2}$

d) 11 and -1

SECTION 2

2.1a Solve by completing the square: $x^2 - 6x - 5 = 0$

a) $3 + \sqrt{5}$ and $3 - \sqrt{5}$

b) $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$

c) $3 + \sqrt{14}$ and $3 - \sqrt{14}$

d) $-3 + \sqrt{14}$ and $-3 - \sqrt{14}$

2.1b Solve by completing the square: $x^2 - 5x = 2$

a) $\frac{5 + \sqrt{33}}{2}$ and $\frac{5 - \sqrt{33}}{2}$

b) $\frac{5 + 4\sqrt{2}}{4}$ and $\frac{5 - 4\sqrt{2}}{4}$

c) $\frac{5 - 2\sqrt{2}}{2}$ and $\frac{5 + 2\sqrt{2}}{2}$

d) $\frac{5 + \sqrt{37}}{4}$ and $\frac{5 - \sqrt{37}}{4}$

2.1c Solve by completing the square: $2x^2 - 4x - 5 = 0$

a) $\frac{2 + \sqrt{14}}{2}$ and $\frac{2 - \sqrt{14}}{2}$

b) $1 - \sqrt{6}$ and $1 + \sqrt{6}$

c) $\frac{2 + \sqrt{10}}{2}$ and $\frac{2 - \sqrt{10}}{2}$

d) $\frac{1 + \sqrt{6}}{2}$ and $\frac{1 - \sqrt{6}}{2}$

2.1d Solve by completing the square: $3x^2 + 7x = -3$

a) $\frac{-7 + \sqrt{22}}{3}$ and $\frac{-7 - \sqrt{22}}{3}$

b) $\frac{-7 + \sqrt{3}}{3}$ and $\frac{-7 - \sqrt{3}}{3}$

c) $\frac{-7 + 4\sqrt{5}}{3}$ and $\frac{-7 - 4\sqrt{5}}{3}$

d) $\frac{-7 + \sqrt{13}}{6}$ and $\frac{-7 - \sqrt{13}}{6}$

SECTION 3

3.1a Solve by using the quadratic formula: $x^2 + 3x - 7 = 0$

a) $\frac{-3 + \sqrt{19}}{2}$ and $\frac{-3 - \sqrt{19}}{2}$

b) $\frac{-3 + \sqrt{37}}{2}$ and $\frac{-3 + \sqrt{37}}{2}$

c) $\frac{-3 + \sqrt{31}}{2}$ and $\frac{-3 - \sqrt{31}}{2}$

d) $\frac{-3 + \sqrt{10}}{2}$ and $\frac{-3 - \sqrt{10}}{2}$

3.1b Solve by using the quadratic formula: $x^2 - 5x = 1$

a) $\frac{5 + \sqrt{21}}{2}$ and $\frac{5 - \sqrt{21}}{2}$

b) $\frac{5 + \sqrt{3}}{2}$ and $\frac{5 - \sqrt{3}}{2}$

c) $\frac{5 + \sqrt{29}}{2}$ and $\frac{5 - \sqrt{29}}{2}$

d) $\frac{5 + 2\sqrt{2}}{2}$ and $\frac{5 - 2\sqrt{2}}{2}$

Review/Test

3.1c Solve by using the quadratic formula: $2x^2 - 3x - 2 = 0$

a) -2 and $\frac{1}{2}$

b) -1 and 4

c) -4 and 1

d) $-\frac{1}{2}$ and 2

3.1d Solve by using the quadratic formula: $3x^2 - 2x = 3$

a) $1 + \sqrt{10}$ and $1 - \sqrt{10}$

b) $2 + \sqrt{10}$ and $2 - \sqrt{10}$

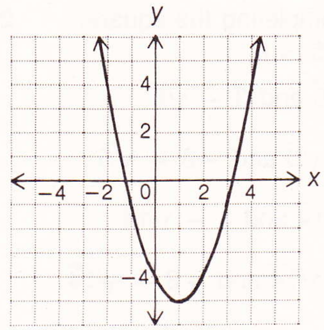
c) $\frac{1 + \sqrt{10}}{3}$ and $\frac{1 - \sqrt{10}}{3}$

d) $\frac{1 + 2\sqrt{10}}{3}$ and $\frac{1 - 2\sqrt{10}}{3}$

SECTION 4

4.1 Graph $y = x^2 - 2x - 3$.

a)



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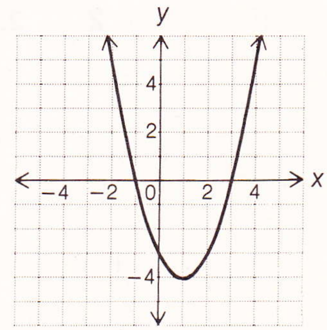

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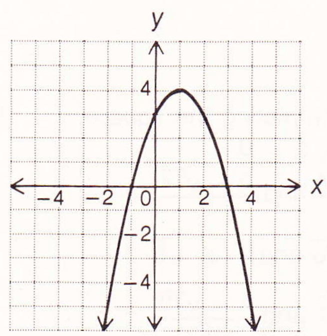
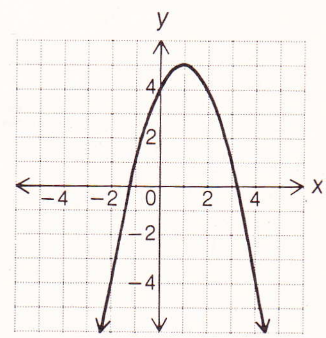


d)



SECTION 5

5.1a The sum of the squares of three consecutive odd integers is 83. Find the middle odd integer.

a) 3 b) 5 c) 7 d) 9

5.1b A jogger ran 7 mi at a constant rate and then reduced the rate by 3 mph. An additional 8 mi was run at the reduced rate. The total time spent jogging the 15 mi was 3 h. Find the rate for the last 8 mi.

a) 4 mph b) 5 mph c) 6 mph d) 7 mph