UNIT 6 Factoring

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List of Objectives

To find the greatest common factor (GCF) of two or more monomials

To factor a monomial from a polynomial

To factor a trinomial of the form $x^2 + bx + c$

To factor completely

To factor a trinomial of the form $ax^2 + bx + c$

To factor completely

To factor the difference of two perfect squares

To factor a perfect square trinomial

To factor a common binomial factor

To factor completely

To solve equations by factoring

To solve application problems

SECTION 1 Monomial Factors

1.1 Objective To find the greatest common factor (GCF) of two or more monomials

The **greatest common factor (GCF)** of two or more integers is the greatest integer which is a factor of all of the integers.

$24 = 2 \cdot 2 \cdot 2 \cdot 3$
$60 = 2 \cdot 2 \cdot 3 \cdot 5$
$GCF = 2 \cdot 2 \cdot 3 = 12$

The GCF of two or more monomials is the product of the GCF of the coefficients and the common variable factors.

$6x^3y = 2 \cdot 3 \cdot x \cdot x \cdot x \cdot y$
$8x^2y^2 = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot y \cdot y$
$GCF = 2 \cdot x \cdot x \cdot y = 2x^2y$

Note that the exponent of each variable in the GCF is the same as the *smallest* exponent of that variable in either of the monomials.

The GCF of $6x^3y$ and $8x^2y^2$ is $2x^2y$.

Find the GCF of $12a^4b$ and $18a^2b^2c$.

$12a^4b = 2 \cdot 2 \cdot 3 \cdot a^4 \cdot b$
$18a^2b^2c = 2 \cdot 3 \cdot 3 \cdot a^2 \cdot b^2 \cdot c$
$GCF = 2 \cdot 3 \cdot a^2 \cdot b = 6a^2b$

The common variable factors are a^2 and b. c is not a common variable factor.

Example 1

Find the GCF of $4x^4y$ and $18x^2y^6$.

Solution

$4x^4y = 2 \cdot 2 \cdot x^4 \cdot y$
$18x^2y^6 = 2 \cdot 3 \cdot 3 \cdot x^2 \cdot y^6$
The GCF is $2x^2y$.

Example 2

Find the GCF of $12x^3y^6$ and $15x^2y^3$.

Your solution

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1.2 Objective To factor a monomial from a polynomial

The Distributive Property is used to multiply factors of a polynomial.
To **factor** a polynomial means to write the polynomial as a product of other polynomials.

Multiply →
Factors $2x(x + 5)$ **Polynomial** $2x^2 + 10x$
← Factor

In the example above, $2x$ is the GCF of the terms $2x^2$ and $10x$. It is a **common monomial factor** of the terms. $x + 5$ is a **binomial factor** of $2x^2 + 10x$.

Factor $5x^3 - 35x^2 + 10x$.

Find the GCF of the terms of the polynomial.

$5x^3 = 5 \cdot x^3$
$35x^2 = 5 \cdot 7 \cdot x^2$
$10x = 2 \cdot 5 \cdot x$
The GCF is $5x$.

Divide each term of the polynomial by the GCF.

$\frac{5x^3}{5x} = x^2 \quad \frac{-35x^2}{5x} = -7x \quad \frac{10x}{5x} = 2$ Do this step mentally.

Use the quotients to rewrite the polynomial, expressing each term as a product with the GCF as one of the factors.

$5x^3 - 35x^2 + 10x = 5x(x^2) + 5x(-7x) + 5x(2)$

Use the Distributive Property to write the polynomial as a product of factors.

$= 5x(x^2 - 7x + 2)$

Example 3

Factor $8x^2 + 2xy$.

Solution

$8x^2 = 2 \cdot 2 \cdot 2 \cdot x^2$
$2xy = 2 \cdot x \cdot y$
The GCF is $2x$.

$8x^2 + 2xy = 2x(4x) + 2x(y) = 2x(4x + y)$

Example 4

Factor $14a^2 - 21a^4b$.

Your solution

Example 5

Factor $16x^2y + 8x^4y^2 - 12x^4y^5$.

Solution

$16x^2y = 2 \cdot 2 \cdot 2 \cdot 2 \cdot x^2 \cdot y$
$8x^4y^2 = 2 \cdot 2 \cdot 2 \cdot x^4 \cdot y^2$
$12x^4y^5 = 2 \cdot 2 \cdot 3 \cdot x^4 \cdot y^5$
The GCF is $4x^2y$.

$16x^2y + 8x^4y^2 - 12x^4y^5 =$
$4x^2y(4) + 4x^2y(2x^2y) + 4x^2y(-3x^2y^4) =$
$4x^2y(4 + 2x^2y - 3x^2y^4)$

Example 6

Factor $6x^4y^2 - 9x^3y^2 + 12x^2y^4$.

Your solution

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1.1 Exercises

Find the greatest common factor.

1. x^7, x^3
2. y^6, y^{12}
3. x^2y^4, xy^6
4. a^5b^3, a^3b^8
5. $x^2y^4z^6, xy^8z^2$
6. ab^2c^3, a^3b^2c
7. $a^3b^2c^3, ab^4c^3$
8. x^3y^2z, x^4yz^5
9. $3x^4, 12x^2$
10. $12x, 30x^2$
11. $16a^3, 18a$
12. $8y^3, 12y^6$
13. $14a^3, 49a^7$
14. $12y^2, 27y^4$
15. $3x^2y^2, 5ab^2$
16. $8x^2y^3, 7ab^4$
17. $9a^2b^4, 24a^4b^2$
18. $15a^4b^2, 9ab^5$
19. $ab^3, 4a^2b, 12a^2b^3$
20. $12x^2y, x^4y, 16x$
21. $2x^2y, 4xy, 8x$
22. $16x^2, 8x^4y^2, 12xy$
23. $3x^2y^2, 6x, 9x^3y^3$
24. $4a^2b^3, 8a^3, 12ab^4$

1.2 Exercises

Factor.

25. $5a + 5$
26. $7b - 7$
27. $16 - 8a^2$
28. $12 + 12y^2$
29. $8x + 12$
30. $16a - 24$
31. $30a - 6$
32. $20b + 5$
33. $7x^2 - 3x$
34. $12y^2 - 5y$
35. $3a^2 + 5a^5$
36. $9x - 5x^2$

Factor.

37. $14y^2 + 11y$

38. $6b^3 - 5b^2$

39. $2x^4 - 4x$

40. $3y^4 - 9y$

41. $10x^4 - 12x^2$

42. $12a^5 - 32a^2$

43. $8a^8 - 4a^5$

44. $16y^4 - 8y^7$

45. $x^2y^2 - xy$

46. $a^2b^2 + ab$

47. $3x^2y^4 - 6xy$

48. $12a^2b^5 - 9ab$

49. $x^2y - xy^3$

50. $a^2b + a^4b^2$

51. $2a^5b + 3xy^3$

52. $5x^2y - 7ab^3$

53. $6a^2b^3 - 12b^2$

54. $8x^2y^3 - 4x^2$

55. $a^3 - 3a^2 + 5a$

56. $b^3 - 5b^2 - 7b$

57. $5x^2 - 15x + 35$

58. $8y^2 - 12y + 32$

59. $3x^3 + 6x^2 + 9x$

60. $5y^3 - 20y^2 + 10y$

61. $2x^4 - 4x^3 + 6x^2$

62. $3y^4 - 9y^3 - 6y^2$

63. $2x^3 + 6x^2 - 14x$

64. $3y^3 - 9y^2 + 24y$

65. $2y^5 - 3y^4 + 7y^3$

66. $6a^5 - 3a^3 - 2a^2$

67. $x^3y - 3x^2y^2 + 7xy^3$

68. $2a^2b - 5a^2b^2 + 7ab^2$

69. $5y^3 + 10y^2 - 25y$

70. $4b^5 + 6b^3 - 12b$

71. $3a^2b^2 - 9ab^2 + 15b^2$

72. $8x^2y^2 - 4x^2y + x^2$

SECTION 2 Factoring Polynomials of the Form $x^2 + bx + c$

2.1 Objective To factor a trinomial of the form $x^2 + bx + c$

Trinomials of the form $x^2 + bx + c$, where b and c are integers, are shown at the right.

$x^2 + 8x + 12$, $b = 8$, $c = 12$
$x^2 - 7x + 12$, $b = -7$, $c = 12$
$x^2 - 2x - 15$, $b = -2$, $c = -15$

To **factor** a trinomial of this form means to express the trinomial as the product of two binomials.

Trinomials expressed as the product of binomials are shown at the right.

$x^2 + 8x + 12 = (x + 6)(x + 2)$
$x^2 - 7x + 12 = (x - 3)(x - 4)$
$x^2 - 2x - 15 = (x + 3)(x - 5)$

The method by which factors of a trinomial are found is based upon FOIL. Consider the following binomial products, noting the relationship between the constant terms of the binomials and the terms of the trinomials.

Signs in the binomials are the same

$(x + 6)(x + 2) = x^2 + 2x + 6x + (6)(2) = x^2 + 8x + 12$
sum of 6 and 2 → $8x$
product of 6 and 2 → 12

$(x - 3)(x - 4) = x^2 - 4x - 3x + (-3)(-4) = x^2 - 7x + 12$
sum of -3 and -4 → $-7x$
product of -3 and -4 → 12

Signs in the binomials are opposite

$(x + 3)(x - 5) = x^2 - 5x + 3x + (3)(-5) = x^2 - 2x - 15$
sum of 3 and -5 → $-2x$
product of 3 and -5 → -15

$(x - 4)(x + 6) = x^2 + 6x - 4x + (-4)(6) = x^2 + 2x - 24$
sum of -4 and 6 → $2x$
product of -4 and 6 → -24

Important Relationships

1. When the constant term of the trinomial is positive, the constant terms of the binomials have the same sign. They are both positive when the coefficient of the x term in the trinomial is positive. They are both negative when the coefficient of the x term in the trinomial is negative.

2. When the constant term of the trinomial is negative, the constant terms of the binomials have opposite signs.

3. In the trinomial, the coefficient of x is the sum of the constant terms of the binomials.

4. In the trinomial, the constant term is the product of the constant terms of the binomials.

The following trinomial factoring patterns help to summarize the relationships stated above.

Trinomial	Factoring Pattern
$x^2 + bx + c$	$(x + \blacksquare)(x + \blacksquare)$
$x^2 - bx + c$	$(x - \blacksquare)(x - \blacksquare)$
$x^2 + bx - c$	$(x + \blacksquare)(x - \blacksquare)$
$x^2 - bx - c$	$(x + \blacksquare)(x - \blacksquare)$

Factor $x^2 + 7x + 10$.

The constant term is positive.
The coefficient of x is positive.
The binomial constants will be positive.

$(x + \blacksquare)(x + \blacksquare)$

Find two positive factors of 10 whose sum is 7.

Factors	Sum
+1, +10	11
+2, +5	7

Write the factors of the trinomial.

$(x + 2)(x + 5)$

Check:
$(x + 2)(x + 5) = x^2 + 5x + 2x + 10$
$= x^2 + 7x + 10$

Factor $x^2 - 8x - 9$.

The constant term is negative.
The signs of the binomial constants will be opposites.

$(x + \blacksquare)(x - \blacksquare)$

Find two factors of 9, one of which is positive and one of which is negative, whose sum is -8.

Once the sum of -8 is found, other factors need not be tried.

Factors	Sum
−1, +9	8
+1, −9	−8
+3, −3	0

Write the factors of the trinomial.

$(x + 1)(x - 9)$

Check:
$(x + 1)(x - 9) = x^2 - 9x + x - 9$
$= x^2 - 8x - 9$

When only integers are used, some trinomials do not factor. For example, to factor $x^2 + 5x + 3$, it would be necessary to find two positive integers whose product is 3 and whose sum is 5. This is not possible, since the only positive factors of 3 are 1 and 3, and the sum of 1 and 3 is 4. This trinomial is **irreducible over the integers.** Binomials of the form $x + a$ or $x - a$ are also irreducible over the integers.

Example 1

Factor $x^2 - 8x + 15$.

Solution

$(x - ■)(x - ■)$

Factors	Sum
$-1, -15$	-16
$-3, -5$	-8

$(x - 3)(x - 5)$

$x^2 - 8x + 15 = (x - 3)(x - 5)$

Example 2

Factor $x^2 - 9x + 20$.

Your solution

Example 3

Factor $x^2 + 6x - 27$.

Solution

$(x + ■)(x - ■)$

Factors	Sum
$+1, -27$	-26
$-1, +27$	26
$+3, -9$	-6
$-3, +9$	6

$(x + 9)(x - 3)$

$x^2 + 6x - 27 = (x + 9)(x - 3)$

Example 4

Factor $x^2 + 3x - 18$.

Your solution

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2.2 Objective To factor completely

A polynomial is factored completely when it is written as a product of factors which are irreducible over the integers.

Factor $3x^3 + 15x^2 + 18x$.

Find the GCF of the terms of the polynomial.

The GCF is $3x$.

Factor out the GCF.

$3x^3 + 15x^2 + 18x =$

$3x(x^2) + 3x(5x) + 3x(6) =$ (Do this step mentally.)

$3x(x^2 + 5x + 6)$

Factor the trinomial.
Find two positive factors of 6 whose sum is 5.

$3x(x + ■)(x + ■)$

Factors	Sum
$+1, +6$	7
$+2, +3$	5

Write the product of the GCF and the factors of the trinomial.

$3x(x + 2)(x + 3)$

Check: $3x(x + 2)(x + 3) =$
$3x(x^2 + 3x + 2x + 6) =$
$3x(x^2 + 5x + 6)$
$3x^3 + 15x^2 + 18x$

Factor $x^2 + 9xy + 20y^2$.

The terms have no common factor.
There are two variables.
Find two positive factors of 20 whose sum is 9.

$(x + \blacksquare y)(x + \blacksquare y)$

Factors	Sum
+1, +20	21
+2, +10	12
+4, +5	9

Write the factors of the trinomial.

$(x + 4y)(x + 5y)$

Check: $(x + 4y)(x + 5y) =$
$x^2 + 5xy + 4xy + 20y^2 =$
$x^2 + 9xy + 20y^2$

Example 5

Factor $2x^2y + 12xy - 14y$.

Solution

The GCF is $2y$.

$2x^2y + 12xy - 14y = 2y(x^2 + 6x - 7)$

Factor the trinomial.

$2y(x + \blacksquare)(x - \blacksquare)$

Factors	Sum
+1, −7	−6
−1, +7	6

$2y(x + 7)(x - 1)$

$2x^2y + 12xy - 14y = 2y(x + 7)(x - 1)$

Example 6

Factor $3a^2b - 18ab - 81b$.

Your solution

Example 7

Factor $4x^2 - 40xy + 84y^2$.

Solution

The GCF is 4.

$4x^2 - 40xy + 84y^2 = 4(x^2 - 10xy + 21y^2)$

Factor the trinomial.

$4(x - \blacksquare y)(x - \blacksquare y)$

Factors	Sum
−1, −21	−22
−3, −7	−10

$4(x - 3y)(x - 7y)$

$4x^2 - 40xy + 84y^2 = 4(x - 3y)(x - 7y)$

Example 8

Factor $3x^2 - 9xy - 12y^2$.

Your solution

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2.1 Exercises

Factor.

1. $x^2 + 3x + 2$
2. $x^2 + 5x + 6$
3. $x^2 - x - 2$
4. $x^2 + x - 6$
5. $a^2 + a - 12$
6. $a^2 - 2a - 35$
7. $a^2 - 3a + 2$
8. $a^2 - 5a + 4$
9. $a^2 + a - 2$
10. $a^2 - 2a - 3$
11. $b^2 - 6b + 9$
12. $b^2 + 8b + 16$
13. $b^2 + 7b - 8$
14. $y^2 - y - 6$
15. $y^2 + 6y - 55$
16. $z^2 - 4z - 45$
17. $y^2 - 5y + 6$
18. $y^2 - 8y + 15$
19. $z^2 - 14z + 45$
20. $z^2 - 14z + 49$
21. $z^2 - 12z - 160$
22. $p^2 + 2p - 35$
23. $p^2 + 12p + 27$
24. $p^2 - 6p + 8$
25. $x^2 + 20x + 100$
26. $x^2 + 18x + 81$
27. $b^2 + 9b + 20$
28. $b^2 + 13b + 40$
29. $x^2 - 11x - 42$
30. $x^2 + 9x - 70$
31. $b^2 - b - 20$
32. $b^2 + 3b - 40$
33. $y^2 - 14y - 51$
34. $y^2 - y - 72$
35. $p^2 - 4p - 21$
36. $p^2 + 16p + 39$

Factor.

37. $y^2 - 8y + 32$

38. $y^2 - 9y + 81$

39. $x^2 - 20x + 75$

40. $p^2 + 24p + 63$

41. $x^2 - 15x + 56$

42. $x^2 + 21x + 38$

43. $x^2 + x - 56$

44. $x^2 + 5x - 36$

45. $a^2 - 21a - 72$

46. $a^2 - 7a - 44$

47. $a^2 - 15a + 36$

48. $a^2 - 21a + 54$

49. $z^2 - 9z - 136$

50. $z^2 + 14z - 147$

51. $c^2 - c - 90$

52. $c^2 - 3c - 180$

53. $z^2 + 15z + 44$

54. $p^2 + 24p + 135$

55. $c^2 + 19c + 34$

56. $c^2 + 11c + 18$

57. $x^2 - 4x - 96$

58. $x^2 + 10x - 75$

59. $x^2 - 22x + 112$

60. $x^2 + 21x - 100$

61. $b^2 + 8b - 105$

62. $b^2 - 22b + 72$

63. $a^2 - 9a - 36$

64. $a^2 + 42a - 135$

65. $b^2 - 23b + 102$

66. $b^2 - 25b + 126$

67. $a^2 + 27a + 72$

68. $z^2 + 24z + 144$

69. $x^2 + 25x + 156$

70. $x^2 - 29x + 100$

71. $x^2 - 10x - 96$

72. $x^2 + 9x - 112$

2.2 Exercises

Factor.

73. $2x^2 + 6x + 4$
74. $3x^2 + 15x + 18$
75. $3a^2 + 3a - 18$
76. $4x^2 - 4x - 8$
77. $ab^2 + 2ab - 15a$
78. $ab^2 + 7ab - 8a$
79. $xy^2 - 5xy + 6x$
80. $xy^2 + 8xy + 15x$
81. $z^3 - 7z^2 + 12z$
82. $2a^3 + 6a^2 + 4a$
83. $3y^3 - 15y^2 + 18y$
84. $4y^3 + 12y^2 - 72y$
85. $3x^2 + 3x - 36$
86. $2x^3 - 2x^2 + 4x$
87. $5z^2 - 15z - 140$
88. $6z^2 + 12z - 90$
89. $2a^3 + 8a^2 - 64a$
90. $3a^3 - 9a^2 - 54a$
91. $x^2 - 5xy + 6y^2$
92. $x^2 + 4xy - 21y^2$
93. $a^2 - 9ab + 20b^2$
94. $a^2 - 15ab + 50b^2$
95. $x^2 - 3xy - 28y^2$
96. $s^2 + 2st - 48t^2$
97. $y^2 - 15yz - 41z^2$
98. $y^2 + 85yz + 36z^2$
99. $z^4 - 12z^3 + 35z^2$
100. $z^4 + 2z^3 - 80z^2$
101. $b^4 - 22b^3 + 120b^2$
102. $b^4 - 3b^3 - 10b^2$
103. $2y^4 - 26y^3 - 96y^2$
104. $3y^4 + 54y^3 + 135y^2$
105. $x^4 + 7x^3 - 8x^2$
106. $x^4 - 11x^3 - 12x^2$
107. $4x^2y + 20xy - 56y$
108. $3x^2y - 6xy - 45y$

Factor.

109. $8y^2 - 32y + 24$

110. $10y^2 - 100y + 90$

111. $c^3 + 13c^2 + 30c$

112. $c^3 + 18c^2 - 40c$

113. $3x^3 - 36x^2 + 81x$

114. $4x^3 + 4x^2 - 24x$

115. $x^2 - 8xy + 15y^2$

116. $y^2 - 7xy - 8x^2$

117. $a^2 - 13ab + 42b^2$

118. $y^2 + 4yz - 21z^2$

119. $y^2 + 8yz + 7z^2$

120. $y^2 - 16yz + 15z^2$

121. $3x^2y + 60xy - 63y$

122. $4x^2y - 68xy - 72y$

123. $3x^3 + 3x^2 - 36x$

124. $4x^3 + 12x^2 - 160x$

125. $4z^3 + 32z^2 - 132z$

126. $5z^3 - 50z^2 - 120z$

127. $4x^3 + 8x^2 - 12x$

128. $5x^3 + 30x^2 + 40x$

129. $5p^2 + 25p - 420$

130. $4p^2 - 28p - 480$

131. $p^4 + 9p^3 - 36p^2$

132. $p^4 + p^3 - 56p^2$

133. $t^2 - 12ts + 35s^2$

134. $a^2 - 10ab + 25b^2$

135. $a^2 - 8ab - 33b^2$

136. $x^2 + 4xy - 60y^2$

137. $5x^4 - 30x^3 + 40x^2$

138. $6x^3 - 6x^2 - 120x$

139. $15ab^2 + 45ab - 60a$

140. $20a^2b - 100ab + 120b$

141. $3yx^2 + 36yx - 135y$

142. $4yz^2 - 52yz + 88y$

SECTION 3 Factoring Polynomials of the Form $ax^2 + bx + c$

3.1 Objective To factor a trinomial of the form $ax^2 + bx + c$

Trinomials of the form $ax^2 + bx + c$, where a is a positive integer and b and c are integers, are shown at the right.

$3x^2 - x + 4$, $a = 3$, $b = -1$, $c = 4$

$6x^2 + 8x - 6$, $a = 6$, $b = 8$, $c = -6$

To factor a trinomial of this form, a trial-and-error method is used. Trial factors are written, using the factors of a and c to write the binomials. Then FOIL is used to check for b, the coefficient of the middle term.

To reduce the number of trial factors which must be considered, remember the following.

1. Use the signs of the constant and the coefficient of x in the trinomial to determine the signs of the terms in the binomial factors.

Trinomial	Factoring Pattern		
$ax^2 + bx + c$	$(\blacksquare x + \blacksquare)(\blacksquare x + \blacksquare)$		
$ax^2 - bx + c$	$(\blacksquare x - \blacksquare)(\blacksquare x - \blacksquare)$		
$ax^2 - bx - c$	$(\blacksquare x + \blacksquare)(\blacksquare x - \blacksquare)$	or	$(\blacksquare x - \blacksquare)(\blacksquare x + \blacksquare)$
$ax^2 + bx - c$	$(\blacksquare x + \blacksquare)(\blacksquare x - \blacksquare)$	or	$(\blacksquare x - \blacksquare)(\blacksquare x + \blacksquare)$

2. If the terms of the trinomial do not have a common factor, then the two terms in either one of the binomial factors will not have a common factor.

Factor $2x^2 - 7x + 3$.

The terms have no common factor.
The constant term is positive.
The coefficient of x is negative.
The binomial constants will be negative.

$(\blacksquare x - \blacksquare)(\blacksquare x - \blacksquare)$

Write the factors of 2 (the coefficient of x^2). These factors will be the coefficients of the x terms in the binomial factors.

Factors of 2: 1, 2

Write the negative factors of 3 (the constant term). These factors will be the constants in the binomial factors.

Factors of 3: -1, -3

Write trial factors. Writing the 1 when it is the coefficient of x may be helpful. Use the Outer and Inner products of FOIL to determine the middle term of the trinomial.

Trial Factors	Middle Term
$(1x - 1)(2x - 3)$	$-3x - 2x = -5x$
$(1x - 3)(2x - 1)$	$-x - 6x = -7x$

Write the factors of the trinomial.

$(x - 3)(2x - 1)$

Check: $(x - 3)(2x - 1) =$
$2x^2 - x - 6x + 3 =$
$2x^2 - 7x + 3$

Factor $6x^2 - x - 2$.

The terms have no common factor. The constant term is negative. The signs of the binomial constants will be opposites.

$(\square x + \square)(\square x - \square)$

or

$(\square x - \square)(\square x + \square)$

Write the factors of 6. These factors will be the coefficients of the x terms in the binomial factors.

Factors of 6: 1, 6
2, 3

Write the factors of -2. These factors will be the constants in the binomial factors.

Factors of -2: $-1, +2$
$+1, -2$

Write the trial factors.
Use the Outer and Inner terms of FOIL to determine the middle term of the trinomial.
It is not necessary to test trial factors which have a common factor. For example, $6x + 2$ need not be tested because it has a common factor of 2. Once a trial solution has the correct middle term, other trial factors need not be tried.

Trial Factors	Middle Term
$(1x - 1)(6x + 2)$	Common factor
$(1x + 2)(6x - 1)$	$-x + 12x = 11x$
$(1x + 1)(6x - 2)$	Common factor
$(1x - 2)(6x + 1)$	$x - 12x = -11x$
$(2x - 1)(3x + 2)$	$4x - 3x = x$
$(2x + 2)(3x - 1)$	Common factor
$(2x + 1)(3x - 2)$	$-4x + 3x = -x$
$(2x - 2)(3x + 1)$	Common factor

Write the factors of the trinomial.

$(2x + 1)(3x - 2)$

Check: $(2x + 1)(3x - 2) =$
$6x^2 - 4x + 3x - 2 =$
$6x^2 - x - 2$

Example 1

Factor $3x^2 + x - 2$.

Solution

$(\square x + \square)(\square x - \square)$ or $(\square x - \square)(\square x + \square)$

Factors of 3: 1, 3

Factors of -2: $+1, -2$
$-1, +2$

Trial Factors	Middle Term
$(1x + 1)(3x - 2)$	$-2x + 3x = x$
$(1x - 2)(3x + 1)$	$x - 6x = -5x$
$(1x - 1)(3x + 2)$	$2x - 3x = -x$
$(1x + 2)(3x - 1)$	$-x + 6x = 5x$

$(x + 1)(3x - 2)$

$3x^2 + x - 2 = (x + 1)(3x - 2)$

Example 2

Factor $2x^2 - x - 3$.

Your solution

$(2x - 3)(x + 1)$
$3x^2 + 2x - 3x - 3$
$3x - x - 3$

Solution on p. 466

3.2 Objective To factor completely

Factor $3x^3 - 23x^2 + 14x$.

Find the GCF of the terms of the polynomial.

The GCF is x.

Factor out the GCF.

$3x^3 - 23x^2 + 14x = x(3x^2 - 23x + 14)$

Factor the trinomial.

$x(\square x - \square)(\square x - \square)$

Write the factors of 3.

Factors of 3: 1, 3

Write the negative factors of 14.

Factors of 14: $-1, -14$
$-2, -7$

Write trial factors. Writing the 1 when it is the coefficient of x may be helpful.
Determine the middle term of the trinomial.

Trial Factors	Middle Term
$(1x - 1)(3x - 14)$	$-14x - 3x = -17x$
$(1x - 14)(3x - 1)$	$-x - 42x = -43x$
$(1x - 2)(3x - 7)$	$-7x - 6x = -13x$
$(1x - 7)(3x - 2)$	$-2x - 21x = -23x$

Write the product of the GCF and the factors of the trinomial.

$x(x - 7)(3x - 2)$

Check: $x(x - 7)(3x - 2) =$
$x(3x^2 - 2x - 21x + 14) =$
$x(3x^2 - 23x + 14) =$
$3x^3 - 23x^2 + 14x$

Factor $15 - 2x - x^2$.

The terms have no common factor.
The coefficient of x^2 is -1.
The signs of the binomials will be opposites.

$(\square + \square x)(\square - \square x)$
or
$(\square - \square x)(\square + \square x)$

Write the factors of 15.

Factors of 15: 1, 15
3, 5

Write the factors of -1.

Factors of -1: $1, -1$

Write trial factors.
Determine the middle term of the trinomial.

Trial Factors	Middle Term
$(1 + 1x)(15 - 1x)$	$-x + 15x = 14x$
$(1 - 1x)(15 + 1x)$	$x - 15x = -14x$
$(3 + 1x)(5 - 1x)$	$-3x + 5x = 2x$
$(3 - 1x)(5 + 1x)$	$3x - 5x = -2x$

Write the factors of the trinomial.

$(3 - x)(5 + x)$

Check: $(3 - x)(5 + x) = 15 + 3x - 5x - x^2$
$= 15 - 2x - x^2$

Example 3

Factor $2x^2y + 19xy - 10y$.

Solution

The GCF is y.

$2x^2y + 19xy - 10y = y(2x^2 + 19x - 10)$

Factor the trinomial.

$y(\square x + \square)(\square x - \square)$ or $y(\square x - \square)(\square x + \square)$

Factors of 2: 1, 2 Factors of -10: $+1, -10$
$-1, +10$
$+2, -5$
$-2, +5$

Trial Factors	Middle Terms
$(1x + 1)(2x - 10)$	Common factor
$(1x - 10)(2x + 1)$	$x - 20x = -19x$
$(1x - 1)(2x + 10)$	Common factor
$(1x + 10)(2x - 1)$	$-x + 20x = 19x$
$(1x + 2)(2x - 5)$	$-5x + 4x = -x$
$(1x - 5)(2x + 2)$	Common factor
$(1x - 2)(2x + 5)$	$5x - 4x = x$
$(1x + 5)(2x - 2)$	Common factor

$y(x + 10)(2x - 1)$

$2x^2y + 19xy - 10y = y(x + 10)(2x - 1)$

Example 4

Factor $4a^2b^2 + 26a^2b - 14a^2$.

Your solution

Example 5

Factor $12x - 32x^2 - 12x^3$.

Solution

The GCF is $4x$.

$12x - 32x^2 - 12x^3 = 4x(3 - 8x - 3x^2)$

Factor the trinomial.

$4x(\square + \square x)(\square - \square x)$ or $4x(\square - \square x)(\square + \square x)$

Factors of 3: 1, 3 Factors of -3: $+1, -3$
$-1, +3$

Trial Factors	Middle Term
$(1 + 1x)(3 - 3x)$	Common factor
$(1 - 3x)(3 + 1x)$	$x - 9x = -8x$
$(1 - 1x)(3 + 3x)$	Common factor
$(1 + 3x)(3 - 1x)$	$-x + 9x = 8x$

$4x(1 - 3x)(3 + x)$

$12x - 32x^2 - 12x^3 = 4x(1 - 3x)(3 + x)$

Example 6

Factor $12y + 12y^2 - 45y^3$.

Your solution

Solutions on p. 466

3.1 Exercises

Factor.

1. $2x^2 + 3x + 1$
2. $5x^2 + 6x + 1$
3. $2y^2 + 7y + 3$
4. $3y^2 + 7y + 2$
5. $2a^2 - 3a + 1$
6. $3a^2 - 4a + 1$
7. $2b^2 - 11b + 5$
8. $3b^2 - 13b + 4$
9. $2x^2 + x - 1$
10. $4x^2 - 3x - 1$
11. $2x^2 - 5x - 3$
12. $3x^2 + 5x - 2$
13. $2t^2 - t - 10$
14. $2t^2 + 5t - 12$
15. $3p^2 - 16p + 5$
16. $6p^2 + 5p + 1$
17. $12y^2 - 7y + 1$
18. $6y^2 - 5y + 1$
19. $6z^2 - 7z + 3$
20. $9z^2 + 3z + 2$
21. $6t^2 - 11t + 4$
22. $10t^2 + 11t + 3$
23. $8x^2 + 33x + 4$
24. $7x^2 + 50x + 7$
25. $5x^2 - 62x - 7$
26. $9x^2 - 13x - 4$
27. $12y^2 + 19y + 5$
28. $5y^2 - 22y + 8$
29. $7a^2 + 47a - 14$
30. $11a^2 - 54a - 5$
31. $3b^2 - 16b + 16$
32. $6b^2 - 19b + 15$
33. $2z^2 - 27z - 14$
34. $4z^2 + 5z - 6$
35. $3p^2 + 22p - 16$
36. $7p^2 + 19p + 10$

Factor.

37. $6x^2 - 17x + 12$

38. $15x^2 - 19x + 6$

39. $5b^2 + 33b - 14$

40. $8x^2 - 30x + 25$

41. $6a^2 + 7a - 24$

42. $14a^2 + 15a - 9$

43. $4z^2 + 11z + 6$

44. $6z^2 - 25z + 14$

45. $22p^2 + 51p - 10$

46. $14p^2 - 41p + 15$

47. $8y^2 + 17y + 9$

48. $12y^2 - 145y + 12$

49. $18t^2 - 9t - 5$

50. $12t^2 + 28t - 5$

51. $6b^2 + 71b - 12$

52. $8b^2 + 65b + 8$

53. $9x^2 + 12x + 4$

54. $25x^2 - 30x + 9$

55. $6b^2 - 13b + 6$

56. $20b^2 + 37b + 15$

57. $33b^2 + 34b - 35$

58. $15b^2 - 43b + 22$

59. $18y^2 - 39y + 20$

60. $24y^2 + 41y + 12$

61. $15a^2 + 26a - 21$

62. $6a^2 + 23a + 21$

63. $8y^2 - 26y + 15$

64. $18y^2 - 27y + 4$

65. $8z^2 + 2z - 15$

66. $10z^2 + 3z - 4$

67. $15x^2 - 82x + 24$

68. $13z^2 + 49z - 8$

69. $10z^2 - 29z + 10$

70. $15z^2 - 44z + 32$

71. $36z^2 + 72z + 35$

72. $16z^2 + 8z - 35$

3.2 Exercises

Factor.

73. $4x^2 + 6x + 2$
74. $12x^2 + 33x - 9$
75. $15y^2 - 50y + 35$
76. $30y^2 + 10y - 20$
77. $2x^3 - 11x^2 + 5x$
78. $2x^3 - 3x^2 - 5x$
79. $3a^2b - 16ab + 16b$
80. $2a^2b - ab - 21b$
81. $3z^2 + 95z + 10$
82. $8z^2 - 36z + 1$
83. $3x^2 + xy - 2y^2$
84. $6x^2 + 10xy + 4y^2$
85. $3a^2 + 5ab - 2b^2$
86. $2a^2 - 9ab + 9b^2$
87. $4y^2 - 11yz + 6z^2$
88. $2y^2 + 7yz + 5z^2$
89. $12 - x - x^2$
90. $2 + x - x^2$
91. $28 + 3z - z^2$
92. $15 - 2z - z^2$
93. $8 - 7x - x^2$
94. $12 + 11x - x^2$
95. $9x^2 + 33x - 60$
96. $16x^2 - 16x - 12$
97. $80y^2 - 36y + 4$
98. $24y^2 - 24y - 18$
99. $8z^3 + 14z^2 + 3z$
100. $6z^3 - 23z^2 + 20z$
101. $6x^2y - 11xy - 10y$
102. $8x^2y - 27xy + 9y$
103. $24x^2 - 52x + 24$
104. $60x^2 + 95x + 20$
105. $35a^4 + 9a^3 - 2a^2$
106. $15a^4 + 26a^3 + 7a^2$
107. $15b^2 - 115b + 70$
108. $25b^2 + 35b - 30$

Factor.

109. $3x^2 - 26xy + 35y^2$ **110.** $4x^2 + 16xy + 15y^2$ **111.** $216y^2 - 3y - 3$

112. $360y^2 + 4y - 4$ **113.** $21 - 20x - x^2$ **114.** $18 + 17x - x^2$

115. $15a^2 + 11ab - 14b^2$ **116.** $15a^2 - 31ab + 10b^2$ **117.** $33z - 8z^2 - z^3$

118. $24z + 10z^2 - z^3$ **119.** $10x^3 + 12x^2 + 2x$ **120.** $9x^3 - 39x^2 + 12x$

121. $10t^2 - 5t - 50$ **122.** $16t^2 + 40t - 96$ **123.** $3p^3 - 16p^2 + 5p$

124. $6p^3 + 5p^2 + p$ **125.** $26z^2 + 98z - 24$ **126.** $30z^2 - 87z + 30$

127. $10y^3 - 44y^2 + 16y$ **128.** $14y^3 + 94y^2 - 28y$ **129.** $4yz^3 + 5yz^2 - 6yz$

130. $2yz^3 - 17yz^2 + 8yz$ **131.** $20b^4 + 41b^3 + 20b^2$ **132.** $6b^4 - 13b^3 + 6b^2$

133. $12a^3 + 14a^2 - 48a$ **134.** $42a^3 + 45a^2 - 27a$ **135.** $36p^2 - 9p^3 - p^4$

136. $9x^2y - 30xy^2 + 25y^3$ **137.** $8x^2y - 38xy^2 + 35y^3$

138. $9x^3y - 24x^2y^2 + 16xy^3$ **139.** $9x^3y + 12x^2y + 4xy$

140. $9a^3b - 9a^2b^2 - 10ab^3$ **141.** $2a^3b - 11a^2b^2 + 5ab^3$

SECTION 4 Special Factoring

4.1 Objective To factor the difference of two perfect squares

The product of a term and itself is called a **perfect square.** The exponents of variables of perfect squares are always even numbers.

Term		Perfect Square
2	$2 \cdot 2 =$	4
x	$x \cdot x =$	x^2
$3y^3$	$3y^3 \cdot 3y^3 =$	$9y^6$

The **square root** of a perfect square is one of the two equal factors of the perfect square. "$\sqrt{\ }$", called a radical, is the symbol for square root. To find the exponent of the square root of a variable term, multiply the exponent by $\frac{1}{2}$.

$\sqrt{4} = 2$

$\sqrt{x^2} = x$

$\sqrt{9y^6} = 3y^3$

The difference of two perfect squares is the product of the sum and difference of two terms.

Sum and Difference of Two Terms		**Difference of Two Perfect Squares**
$(a + b)(a - b)$	$=$	$a^2 - b^2$

The factors of the difference of two perfect squares are the sum and difference of the square roots of the perfect squares.

$a^2 + b^2$ is the *sum* of two perfect squares. It is irreducible over the integers.

Factor $x^2 - 16$.

Write $x^2 - 16$ as the difference of two perfect squares.

$x^2 - 16 = x^2 - 4^2$

The factors are the sum and difference of the square roots of the perfect squares.

$= (x + 4)(x - 4)$

Check: $(x + 4)(x - 4) = x^2 - 4x + 4x - 16$
$= x^2 - 16$

Factor $x^2 - 10$.

Since 10 is not a perfect square, $x^2 - 10$ cannot be written as the difference of two perfect squares. $x^2 - 10$ is irreducible over the integers.

Example 1

Factor $16x^2 - y^2$.

Solution

$16x^2 - y^2 = (4x)^2 - y^2 = (4x + y)(4x - y)$

Example 2

Factor $25a^2 - b^2$.

Your solution

Example 3

Factor $z^6 - 25$.

Solution

$z^6 - 25 = (z^3)^2 - 5^2 = (z^3 + 5)(z^3 - 5)$

Example 4

Factor $n^8 - 36$.

Your solution

Solutions on p. 467

4.2 Objective To factor a perfect square trinomial

A perfect square trinomial is the square of a binomial.

Square of a Binomial		**Perfect Square Trinomial**
$(a + b)^2$	$= (a + b)(a + b) =$	$a^2 + 2ab + b^2$
$(a - b)^2$	$= (a - b)(a - b) =$	$a^2 - 2ab + b^2$

In factoring a perfect square trinomial, remember that the terms of the binomial are the square roots of the perfect squares of the trinomial. The sign in the binomial is the sign of the middle term of the trinomial.

Factor $x^2 + 10x + 25$.

Check that the trinomial is a perfect square.

$\sqrt{x^2} = x$ $\quad 2(5x) = 10x$
$\sqrt{25} = 5$

The trinomial is a perfect square.

Write the factors as the square of a binomial.

$(x + 5)^2$

Check: $(x + 5)^2 = (x + 5)(x + 5)$
$= x^2 + 5x + 5x + 25$
$= x^2 + 10x + 25$

Factor $x^2 + 10x - 25$.

Since the constant term is negative, $x^2 + 10x - 25$ is not a perfect square trinomial. $x^2 + 10x - 25$ is irreducible over the integers.

Example 5

Factor $y^2 - 14y + 49$.

Solution

$\sqrt{y^2} = y$ $\quad 2(7y) = 14y$
$\sqrt{49} = 7$

The trinomial is a perfect square.

$y^2 - 14y + 49 = (y - 7)^2$

Example 6

Factor $a^2 + 20a + 100$.

Your solution

Example 7

Factor $9x^2 - 24xy + 16y^2$.

Solution

$\sqrt{9x^2} = 3x$ $\quad 2(3x \cdot 4y) = 24xy$
$\sqrt{16y^2} = 4y$

The trinomial is a perfect square.

$9x^2 - 24xy + 16y^2 = (3x - 4y)^2$

Example 8

Factor $25a^2 - 30ab + 9b^2$.

Your solution

Solutions on p. 467

4.3 Objective To factor a common binomial factor

In the examples at the right, the binomials in parentheses are called binomial factors.

$2a(a + b)^2$

$3xy(x - y)$

The Distributive Property is used to factor a common binomial factor from an expression.

Factor $6(x - 3) + y^2(x - 3)$.

The common binomial factor is $x - 3$. Use the Distributive property to write the expression as a product of factors.

$6(x - 3) + y^2(x - 3) =$

$(x - 3)(6 + y^2)$

Factor $2x(a - b) + 5(b - a)$.

Rewrite the expression as a difference of terms which have a common factor. Note that $(b - a) = (-a + b) = -(a - b)$.

$2x(a - b) + 5(b - a)$

$2x(a - b) + 5[-(a - b)]$ Do this step mentally.

$2x(a - b) - 5(a - b)$

Write the expression as a product of factors.

$(a - b)(2x - 5)$

Example 9

Factor $4x(3x - 2) - 7(3x - 2)$.

Solution

$4x(3x - 2) - 7(3x - 2) = (3x - 2)(4x - 7)$

Example 10

Factor $5x(2x + 3) - 4(2x + 3)$.

Your solution

Example 11

Factor $5a(2x - 7) + 2(7 - 2x)$.

Solution

$5a(2x - 7) + 2(7 - 2x) =$

$5a(2x - 7) - 2(2x - 7) = (2x - 7)(5a - 2)$

Example 12

Factor $2y(5x - 2) - 3(2 - 5x)$.

Your solution

Solutions on p. 467

4.4 Objective To factor completely

When factoring a polynomial completely, ask the following questions about the polynomial.

1. Is there a common factor? If so, factor out the common factor.
2. Is the polynomial the difference of two perfect squares? If so, factor.
3. Is the polynomial a perfect square trinomial? If so, factor.
4. Is the polynomial a trinomial which is the product of two binomials? If so, factor.
5. Is each factor irreducible over the integers? If not, factor.

Example 13

Factor $3x^2 - 48$.

Solution

The GCF is 3.
$3x^2 - 48 = 3(x^2 - 16)$
Factor the difference of two perfect squares.
$3(x + 4)(x - 4)$

$3x^2 - 48 = 3(x + 4)(x - 4)$

Example 14

Factor $12x^3 - 75x$.

Your solution

Example 15

Factor $x^2(x - 3) + 4(3 - x)$.

Solution

The common binomial factor is $x - 3$.
$x^2(x - 3) + 4(3 - x) =$
$x^2(x - 3) - 4(x - 3) = (x - 3)(x^2 - 4)$
Factor the difference of two perfect squares.
$(x - 3)(x + 2)(x - 2)$

$x^2(x - 3) + 4(3 - x) = (x - 3)(x + 2)(x - 2)$

Example 16

Factor $a^2(b - 7) + (7 - b)$.

Your solution

Example 17

Factor $4x^2y^2 + 12xy^2 + 9y^2$.

Solution

The GCF is y^2.
$4x^2y^2 + 12xy^2 + 9y^2 = y^2(4x^2 + 12x + 9)$
Factor the perfect square trinomial.
$y^2(2x + 3)^2$

$4x^2y^2 + 12xy^2 + 9y^2 = y^2(2x + 3)^2$

Example 18

Factor $4x^3 + 28x^2 - 120x$.

Your solution

Solutions on p. 467

4.1 Exercises

Factor.

1. $x^2 - 4$
2. $x^2 - 9$
3. $a^2 - 81$
4. $a^2 - 49$
5. $4x^2 - 1$
6. $9x^2 - 16$
7. $x^6 - 9$
8. $y^{12} - 64$
9. $25x^2 - 1$
10. $4x^2 - 1$
11. $1 - 49x^2$
12. $1 - 64x^2$
13. $t^2 + 36$
14. $x^2 + 64$
15. $x^4 - y^2$
16. $b^4 - 16a^2$
17. $9x^2 - 16y^2$
18. $25z^2 - y^2$
19. $x^2y^2 - 4$
20. $a^2b^2 - 25$
21. $16 - x^2y^2$

4.2 Exercises

Factor.

22. $y^2 + 2y + 1$
23. $y^2 + 14y + 49$
24. $a^2 - 2a + 1$
25. $x^2 + 8x - 16$
26. $z^2 - 18z - 81$
27. $x^2 - 12x + 36$
28. $x^2 + 2xy + y^2$
29. $x^2 + 6xy + 9y^2$
30. $4a^2 + 4a + 1$
31. $25x^2 + 10x + 1$
32. $64a^2 - 16a + 1$
33. $9a^2 + 6a + 1$

Factor.

34. $16b^2 + 8b + 1$

35. $4a^2 - 20a + 25$

36. $4b^2 + 28b + 49$

37. $9a^2 - 42a + 49$

38. $25a^2 + 30ab + 9b^2$

39. $4a^2 - 12ab + 9b^2$

40. $49x^2 + 28xy + 4y^2$

41. $4y^2 - 36yz + 81z^2$

42. $64y^2 - 48yz + 9z^2$

4.3 Exercises

Factor.

43. $x(a + b) + 2(a + b)$

44. $a(x + y) + 4(x + y)$

45. $x(b + 2) - y(b + 2)$

46. $a(y - 4) - b(y - 4)$

47. $z(x - 3) - (x - 3)$

48. $a(y + 7) - (y + 7)$

49. $x(b - 2c) + y(b - 2c)$

50. $2x(x - 3) - (x - 3)$

51. $a(x - 2) + 5(2 - x)$

52. $a(x - 7) + b(7 - x)$

53. $b(y - 2) - 2a(y - 2)$

54. $x(a - 3) - 2y(a - 3)$

55. $b(y - 3) + 3(3 - y)$

56. $c(a - 2) - b(2 - a)$

57. $a(x - y) - 2(y - x)$

58. $3(a - b) - x(b - a)$

4.4 Exercises

Factor.

59. $5x^2 - 5$

60. $2x^2 - 18$

61. $x^3 + 4x^2 + 4x$

62. $y^3 - 10y^2 + 25y$

63. $x^4 + 2x^3 - 35x^2$

64. $a^4 - 11a^3 + 24a^2$

65. $5b^2 + 75b + 180$

66. $6y^2 - 48y + 72$

67. $3a^2 + 36a + 10$

68. $5a^2 - 30a + 4$

69. $2x^2y + 16xy - 66y$

70. $3a^2b + 21ab - 54b$

71. $x^3 - 6x^2 - 5x$

72. $b^3 - 8b^2 - 7b$

73. $3y^2 - 36$

74. $3y^2 - 147$

75. $20a^2 + 12a + 1$

76. $12a^2 - 36a + 27$

77. $x^2y^2 - 7xy^2 - 8y^2$

78. $a^2b^2 + 3a^2b^2 - 88a^2$

79. $10a^2 - 5ab - 15b^2$

80. $16x^2 - 32xy + 12y^2$

81. $50 - 2x^2$

82. $72 - 2x^2$

83. $a^2b^2 - 10ab^2 + 25b^2$

84. $a^2b^2 + 6ab^2 + 9b^2$

85. $12a^3b - a^2b^2 - ab^3$

86. $2x^3y - 7x^2y^2 + 6xy^3$

87. $12a^3 - 12a^2 + 3a$

88. $18a^3 + 24a^2 + 8a$

89. $243 + 3a^2$

90. $75 + 27y^2$

91. $12a^3 - 46a^2 + 40a$

92. $24x^3 - 66x^2 + 15x$

93. $4a^3 + 20a^2 + 25a$

94. $2a^3 - 8a^2b + 8ab^2$

Factor.

95. $27a^2b - 18ab + 3b$

96. $a^2b^2 - 6ab^2 + 9b^2$

97. $48 - 12x - 6x^2$

98. $21x^2 - 11x^3 - 2x^4$

99. $x^4 - x^2y^2$

100. $b^4 - a^2b^2$

101. $18a^3 + 24a^2 + 8a$

102. $32xy^2 - 48xy + 18x$

103. $2b + ab - 6a^2b$

104. $20x - 11xy - 3xy^2$

105. $72xy^2 + 48xy + 8x$

106. $4x^2y + 8xy + 4y$

107. $15y^2 - 2xy^2 - x^2y^2$

108. $4x^4 - 38x^3 + 48x^2$

109. $3x^2 - 27y^2$

110. $x^4 - 25x^2$

111. $y^3 - 9y$

112. $a^4 - 16$

113. $15x^4y^2 - 13x^3y^3 - 20x^2y^4$

114. $45y^2 - 42y^3 - 24y^4$

115. $a(2x - 2) + b(2x - 2)$

116. $4a(x - 3) - 2b(x - 3)$

117. $x^2(x - 2) - (x - 2)$

118. $y^2(a - b) - (a - b)$

119. $a(x^2 - 4) + b(x^2 - 4)$

120. $x(a^2 - b^2) - y(a^2 - b^2)$

121. $4(x - 5) - x^2(x - 5)$

122. $y^2(a - b) - 9(a - b)$

123. $x^2(x - 2) + 4(2 - x)$

124. $a(2y^2 - 4) - b(4 - 2y^2)$

SECTION 5 Solving Equations

5.1 Objective To solve equations by factoring

Recall that the Multiplication Property of Zero states that the product of a number and zero is zero.

If a is a real number, then $a \cdot 0 = 0 \cdot a = 0$.

Consider $x \cdot y = 0$. If this is a true equation, then either $x = 0$ or $y = 0$.

Principle of Zero Products

If the product of two factors is zero, then at least one of the factors must be zero.

If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

The Principle of Zero Products is used in solving equations.

Solve: $(x - 2)(x - 3) = 0$

If $(x - 2)(x - 3) = 0$, then $(x - 2) = 0$ or $(x - 3) = 0$.

$$(x - 2)(x - 3) = 0$$

Rewrite each equation in the form *variable = constant*.

$$x - 2 = 0 \qquad x - 3 = 0$$
$$x = 2 \qquad x = 3$$

Write the solution.

The solutions are 2 and 3.

Check:

$$(x - 2)(x - 3) = 0 \qquad (x - 2)(x - 3) = 0$$
$$(2 - 2)(2 - 3) = 0 \qquad (3 - 2)(3 - 3) = 0$$
$$0(-1) = 0 \qquad -1(0) = 0$$
$$0 = 0 \qquad 0 = 0$$

A true equation   A true equation

An equation of the form $ax^2 + bx + c = 0$ is a **quadratic equation.** A quadratic equation is in **standard form** when the polynomial is in descending order and equal to zero.

$$3x^2 + 2x + 1 = 0$$
$$4x^2 - 3x + 2 = 0$$

Solve: $2x^2 + x = 6$

$$2x^2 + x = 6$$

Write the equation in standard form.

$$2x^2 + x - 6 = 0$$

Factor.

$$(2x - 3)(x + 2) = 0$$

Let each factor equal zero (the Principle of Zero Products).

$$2x - 3 = 0 \qquad x + 2 = 0$$

Rewrite each equation in the form *variable = constant*.

$$2x = 3 \qquad x = -2$$
$$x = \frac{3}{2}$$

Write the solution.

The solutions are $\frac{3}{2}$ and -2.

$\frac{3}{2}$ and -2 check as solutions.

Example 1

Solve: $x(x - 3) = 0$

Solution

$x(x - 3) = 0$

$x = 0$ $\qquad$ $x - 3 = 0$

$x = 3$

The solutions are 0 and 3.

Example 2

Solve: $2x(x + 7) = 0$

Your solution

Example 3

Solve: $2x^2 - 50 = 0$

Solution

$2x^2 - 50 = 0$

$2(x^2 - 25) = 0$

$2(x + 5)(x - 5) = 0$

$x + 5 = 0$ $\qquad$ $x - 5 = 0$

$x = -5$ $\qquad$ $x = 5$

The solutions are -5 and 5.

Example 4

Solve: $4x^2 - 9 = 0$

Your solution

Example 5

Solve: $(x - 3)(x - 10) = -10$

Solution

$(x - 3)(x - 10) = -10$ $\qquad$ Write in standard form.

$x^2 - 13x + 30 = -10$

$x^2 - 13x + 40 = 0$

$(x - 8)(x - 5) = 0$

$x - 8 = 0$ $\qquad$ $x - 5 = 0$

$x = 8$ $\qquad$ $x = 5$

The solutions are 8 and 5.

Example 6

Solve: $(x + 2)(x - 7) = 52$

Your solution

Solutions on p. 468

5.2 Objective To solve application problems

Example 7

The sum of the squares of two consecutive positive even integers is equal to 100. Find the two integers.

Strategy

First positive even integer: n
Second positive even integer: $n + 2$

The sum of the square of the first positive even integer and the square of the second positive even integer is 100.

Solution

$n^2 + (n + 2)^2 = 100$

$n^2 + n^2 + 4n + 4 = 100$

$2n^2 + 4n + 4 = 100$

$2n^2 + 4n - 96 = 0$

$2(n^2 + 2n - 48) = 0$

$2(n - 6)(n + 8) = 0$

$n - 6 = 0 \qquad n + 8 = 0$
$n = 6 \qquad n = -8$

Since -8 is not a positive even integer, it is not a solution.

$n = 6$
$n + 2 = 6 + 2 = 8$

The two integers are 6 and 8.

Example 8

The sum of the squares of two consecutive positive integers is 61. Find the two integers.

Your strategy

Your solution

Solution on p. 469

Example 9

A stone is thrown into a well with an initial speed of 4 ft/s. The well is 420 ft deep. How many seconds later will the stone hit the bottom of the well? Use the equation $d = vt + 16t^2$, where d is the distance in feet, v is the initial speed, and t is the time in seconds.

Strategy

To find the time for the stone to drop to the bottom of the well, replace the variables d and v by their given values and solve for t.

Solution

$d = vt + 16t^2$

$420 = 4t + 16t^2$

$0 = -420 + 4t + 16t^2$

$16t^2 + 4t - 420 = 0$

$4(4t^2 + t - 105) = 0$

$4(4t + 21)(t - 5) = 0$

$4t + 21 = 0$ | $t - 5 = 0$

$4t = -21$ | $t = 5$

$t = -\frac{21}{4}$

Since the time cannot be a negative number, $-\frac{21}{4}$ is not a solution.

The time is 5 s.

Example 10

The length of a rectangle is 4 in. longer than twice the width. The area of the rectangle is 96 in.^{2}. Find the length and width of the rectangle.

Your strategy

Your solution

Solution on p. 469

5.1 Exercises

Solve.

1. $(y + 3)(y + 2) = 0$
2. $(y - 3)(y - 5) = 0$
3. $(z - 7)(z - 3) = 0$
4. $(z + 8)(z - 9) = 0$
5. $x(x - 5) = 0$
6. $x(x + 2) = 0$
7. $a(a - 9) = 0$
8. $a(a + 12) = 0$
9. $y(2y + 3) = 0$
10. $t(4t - 7) = 0$
11. $2a(3a - 2) = 0$
12. $4b(2b + 5) = 0$
13. $(b + 2)(b - 5) = 0$
14. $(b - 8)(b + 3) = 0$
15. $x^2 - 81 = 0$
16. $x^2 - 121 = 0$
17. $4x^2 - 49 = 0$
18. $16x^2 - 1 = 0$
19. $9x^2 - 1 = 0$
20. $16x^2 - 49 = 0$
21. $x^2 + 6x + 8 = 0$
22. $x^2 - 8x + 15 = 0$
23. $z^2 + 5z - 14 = 0$
24. $z^2 + z - 72 = 0$
25. $x^2 - 5x + 6 = 0$
26. $x^2 - 3x - 10 = 0$
27. $y^2 + 4y - 21 = 0$
28. $2y^2 - y - 1 = 0$
29. $2a^2 - 9a - 5 = 0$
30. $3a^2 + 14a + 8 = 0$
31. $6z^2 + 5z + 1 = 0$
32. $6y^2 - 19y + 15 = 0$
33. $x^2 - 3x = 0$
34. $a^2 - 5a = 0$
35. $x^2 - 7x = 0$
36. $2a^2 - 8a = 0$

Solve.

37. $a^2 + 5a = -4$

38. $a^2 - 5a = 24$

39. $y^2 - 5y = -6$

40. $y^2 - 7y = 8$

41. $2t^2 + 7t = 4$

42. $3t^2 + t = 10$

43. $3t^2 - 13t = -4$

44. $5t^2 - 16t = -12$

45. $x(x - 12) = -27$

46. $x(x - 11) = 12$

47. $y(y - 7) = 18$

48. $y(y + 8) = -15$

49. $p(p + 3) = -2$

50. $p(p - 1) = 20$

51. $y(y + 4) = 45$

52. $y(y - 8) = -15$

53. $x(x + 3) = 28$

54. $p(p - 14) = 15$

55. $(x + 8)(x - 3) = -30$

56. $(x + 4)(x - 1) = 14$

57. $(y + 3)(y + 10) = -10$

58. $(z - 5)(z + 4) = 52$

59. $(z - 8)(z + 4) = -35$

60. $(z - 6)(z + 1) = -10$

61. $(a + 3)(a + 4) = 72$

62. $(a - 4)(a + 7) = -18$

63. $(2x + 5)(x + 1) = -1$

64. $(z + 3)(z - 10) = -42$

65. $(y + 3)(2y + 3) = 5$

66. $(y + 5)(3y - 2) = -14$

5.2 Application Problems

Solve.

1. The square of a positive number is six more than five times the positive number. Find the number.

2. The square of a negative number is sixteen more than six times the negative number. Find the number.

3. The sum of two numbers is six. The sum of the squares of the two numbers is twenty. Find the two numbers.

4. The sum of two numbers is eight. The sum of the squares of the two numbers is thirty-four. Find the two numbers.

5. The sum of the squares of two consecutive positive integers is eighty-five. Find the two integers.

6. The sum of the squares of two consecutive positive even integers is one hundred. Find the two integers.

7. The sum of two numbers is ten. The product of the two numbers is twenty-one. Find the two numbers.

8. The sum of two numbers is twenty-three. The product of the two numbers is one hundred twenty. Find the two numbers.

9. The square of the sum of a number and three is one hundred forty-four. Find the number.

10. The square of the sum of a number and five is eighty-one. Find the number.

11. The product of two consecutive positive integers is two hundred ten. Find the integers.

12. The product of two consecutive odd positive integers is one hundred forty-three. Find the integers.

Solve.

13. The length of the base of a triangle is four times the height. The area of the triangle is 50 ft^2. Find the base and height of the triangle.

14. The height of a triangle is 3 m more than twice the length of the base. The area of the triangle is 76 m^2. Find the height of the triangle.

15. The length of a rectangle is three times the width. The area is 300 $in.^2$. Find the length and width of the rectangle.

16. The length of a rectangle is two more than twice the width. The area is 312 ft^2. Find the length and width of the rectangle.

17. The length of a rectangle is 5 in. more than twice the width. The area is 75 $in.^2$. Find the length and width of the rectangle.

18. The width of a rectangle is 5 ft less than the length. The area of the rectangle is 176 ft. Find the length and width of the rectangle.

19. The length of each side of a square is extended 2 in. The area of the resulting square is 144 $in.^2$. Find the length of a side of the original square.

20. The length of each side of a square is extended 5 in. The area of the resulting square is 64 $in.^2$. Find the length of a side of the original square.

21. An object is thrown downward, with an initial speed of 16 ft/s, from the top of a building 320 ft high. How many seconds later will the object hit the ground? Use the equation $d = vt + 16t^2$, where d is the distance in feet, v is the initial speed, and t is the time in seconds.

22. An object falls from an airplane that is flying at an altitude of 6400 ft. How many seconds later will the object hit the ground? Use the equation $16t^2 = d$, where d is the distance in feet and t is the time in seconds.

23. The radius of a circle is increased by 3 in., increasing the area by 100 $in.^2$. Find the radius of the original circle. Use 3.14 for π.

24. A circle has a radius of 10 in. Find the increase in area when the radius is increased by 2 in. Use 3.14 for π.

Review/Test

SECTION 1

1.1 Find the GCF of $12a^2b^3$ and $16ab^6$.

1.2 Factor $6x^3 - 8x^2 + 10x$.

SECTION 2

2.1a Factor $p^2 + 5p + 6$.

2.1b Factor $a^2 - 19a + 48$.

2.1c Factor $x^2 + 2x - 15$.

2.1d Factor $x^2 - 9x - 36$.

2.2a Factor $5x^2 - 45x - 15$.

2.2b Factor $2y^4 - 14y^3 - 16y^2$.

SECTION 3

3.1a Factor $2x^2 + 4x - 5$.

3.1b Factor $6x^2 + 19x + 8$.

3.2a Factor $8x^2 + 20x - 48$.

3.2b Factor $6x^2y^2 + 9xy^2 + 12y^2$.

Review/Test

SECTION 4

4.1a Factor $b^2 - 16$.

4.1b Factor $4x^2 - 49y^2$.

4.2a Factor $p^2 + 12p + 36$.

4.2b Factor $4a^2 - 12ab + 9b^2$.

4.3a Factor $a(x - 2) + b(x - 2)$.

4.3b Factor $x(p + 1) - (p + 1)$.

4.4a Factor $3a^2 - 75$.

4.4b Factor $3x^2 + 12xy + 12y^2$.

SECTION 5

5.1a Solve:
$(2a - 3)(a + 7) = 0$

5.1b Solve: $x(x - 8) = -15$

5.2 The length of a rectangle is 3 cm longer than twice the width. The area of the rectangle is 90 cm^2. Find the length and width of the rectangle.

Review/Test

SECTION 1

1.1 Find the GCF of $12x^3y^2$ and $42xy^6$.
a) $12x^3y^6$
b) $6xy^2$
c) $3xy^2$
d) $6x^3y^6$

1.2 Factor $15xy^2 - 20xy^4$.
a) $5xy^2(3 - 4y^2)$
b) $5(3xy^2 - 4xy^4)$
c) $5xy(3y - 4y^3)$
d) $x(15y^2 - 20y^4)$

SECTION 2

2.1a Factor $b^2 + 10b + 21$.
a) $(b + 10)(b + 3)$
b) $(b + 3)(b + 7)$
c) $(b + 21)(b + 10)$
d) $(b + 13)(b - 3)$

2.1b Factor $y^2 - 7y + 6$.
a) $(y + 3)(y + 2)$
b) $(y - 3)(y - 2)$
c) $(y + 6)(y - 1)$
d) $(y - 6)(y - 1)$

2.1c Factor $a^2 + 3a - 18$.
a) $(a - 3)(a + 6)$
b) $(a + 3)(a - 6)$
c) $(a + 9)(a - 2)$
d) $(a - 3)(a - 6)$

2.1d Factor $p^2 - 9p - 10$.
a) $(p + 1)(p + 10)$
b) $(p + 10)(p - 1)$
c) $(p - 10)(p + 1)$
d) $(p - 10)(p - 1)$

2.2a Factor $5x^2 + 15x + 10$.
a) $(5x + 10)(x + 1)$
b) $(x + 2)(5x + 5)$
c) $5(x + 2)(x + 5)$
d) $5(x + 2)(x + 1)$

2.2b Factor $x^2 - 5xy - 14y^2$.
a) $(x - 2y)(x + 7y)$
b) $(x + 2y)(x - 7y)$
c) $(x - 2y)(x - 7y)$
d) $(x + 2y)(x + 7y)$

SECTION 3

3.1a Factor $12x^2 - x - 1$.
a) $(3x + 1)(4x - 1)$
b) $(6x + 1)(2x - 1)$
c) $(3x - 1)(4x + 1)$
d) $(6x - 1)(2x + 1)$

3.1b Factor $9x^2 + 15x - 14$.
a) $(3x - 2)(3x + 7)$
b) $(3x + 2)(3x - 7)$
c) $(3x - 2)(3x - 7)$
d) $(9x - 1)(x + 7)$

3.2a Factor $2a^3 + 7a^2 - 15a$.
a) $(2a^2 - 3a)(a + 5)$
b) $a(2a + 3)(a - 5)$
c) $a(2a - 3)(a + 5)$
d) $a(2a - 3)(a - 5)$

3.2b Factor $18a^3 + 57a^2 + 30a$.
a) $3a(2a + 5)(3a + 2)$
b) $3(2a + 5)(3a + 2)$
c) $3a(3a + 5)(2a + 2)$
d) $3a(6a + 1)(a + 2)$

Review/Test

SECTION 4

4.1a Factor $p^2 - 64$.
a) $(p + 8)(p + 8)$
b) $(p - 8)(p - 8)$
c) $(p - 4)(p + 4)$
d) $(p - 8)(p + 8)$

4.1b Factor $36a^2 - 49b^2$.
a) $(6a - 7b)(6a - 7b)$
b) $(6a - 7b)(6a + 7b)$
c) $(6a + 7b)(6a + 7b)$
d) $(36a - b)(b + 49b)$

4.2a Factor $b^2 - 10b + 25$.
a) $(b - 5)(b + 5)$
b) $(b - 5)^2$
c) $(b + 5)^2$
d) $(b - 10)^2$

4.2b Factor $4x^2 + 28xy + 49y^2$.
a) $(2x + 7y)(2x - 7y)$
b) $(2x - 7y)^2$
c) $(2x + 7y)^2$
d) $(4x + y)(x + 49y)$

4.3a Factor $x(a + 2) + y(a + 2)$.
a) $(a + 2)(x + y)$
b) $(x + a)(y + 2)$
c) $(x - a)(a + 2)$
d) $(a + y)(x + y)$

4.3b Factor $3y(x - 3) - 2(x - 3)$.
a) $(3y + 2)(x - 3)$
b) $(3y - x)(y + 2)$
c) $(3y - 2)(x - 3)$
d) $(3y + 2)(x + 3)$

4.4a Factor $4b^2 - 100$.
a) $(2b - 10)(2b + 10)$
b) $4(b - 5)^2$
c) $4(b + 5)^2$
d) $4(b - 5)(b + 5)$

4.4b Factor $18x^2 - 48xy + 32y^2$.
a) $2(3x - 44)(3x + 44)$
b) $2(3x - 4y)^2$
c) $2(3x + 4y)^2$
d) $(9x + 2y)(2x + 16y)$

SECTION 5

5.1a Solve: $(x + 3)(2x - 5) = 0$
a) The solutions are -3 and $\frac{5}{2}$.
b) The solutions are -3 and $-\frac{5}{2}$.
c) The solutions are 3 and $-\frac{5}{2}$.
d) The solutions are 3 and $\frac{5}{2}$.

5.1b Solve: $3x^2 + 19x - 14 = 0$
a) The solutions are $-\frac{3}{2}$ and 7.
b) The solutions are $\frac{1}{3}$ and 14.
c) The solutions are $\frac{2}{3}$ and -7.
d) The solutions are $-\frac{2}{3}$ and 7.

5.2 The length of the base of a triangle is three times the height. The area of the triangle is 24 in.^{2}. Find the length of the base of the triangle.
a) 12 in. b) 4 in. c) 48 in. d) 16 in.